NOTES ON HARMONIC ANALYSIS PART II: THE FOURIER SERIES

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Abstract. Fourier Series is the second of monographs we present on harmonic analysis. Harmonic analysis is one of the most fascinating areas of research in mathematics. Its centrality in the development of many areas of mathematics such as partial differential equations and integration theory and its many and diverse applications in sciences and engineering fields makes it an attractive field of study and research.

The purpose of these notes is to introduce the basic ideas and theorems of the subject to students of mathematics, physics, or engineering sciences. Our goal is to illustrate the topics with utmost clarity and accuracy, readily understandable by the students or interested readers. Rather than providing just the outlines or sketches of the proofs, we have actually provided the complete proofs of all theorems. This approach will illuminate the necessary steps taken and the machinery used to complete each proof.

The prerequisite for understanding the topics presented is the knowledge of Lebesgue measure and integral. This will provide ample mathematical background for an advanced undergraduate or a graduate student in mathematics.

1. Definitions and important results

Definition 1.1. *The set of all complex numbers of modulus 1 is denoted by*

$$\mathbf{T} = \{z = e^{ix} : x \in R\}.$$

$\mathbf{T}$ is a compact abelian group with binary operation: complex multiplication and topology: open arcs $\{e^{ix} : x \in (a, b)\}$.

Define the periodic function $F(x)$ on R by

$$F(x) = f(e^{ix}), \quad x \in R,$$

where f is a function on $\mathbf{T}$.

Let χ be the identity function on T, i.e.,

$$\chi(e^{ix}) = e^{ix}, \quad x \in R.$$

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Clearly, $F(x) = \cos\, x + i \sin\, x$ satisfies $F(x) = \chi(e^{ix})$ for all $x \in R$.

Definition 1.2.

$$\mathbf{L^p(T)} = \{f \textit{ defined on } T : \int |f|^p d\sigma < \infty\},$$

where

$$\int |f|^p d\sigma = \int_{-\pi}^{\pi} |f(e^{ix})|^p \frac{dx}{2\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(x)|^p dx.$$

Theorem 1.1.

$$\mathbf{L^p(T)} \supset \mathbf{L^r(T)}, \quad \textit{if } p < r, \textit{ that is} \quad \|f\|_p \le \|f\|_r.$$

Proof: Using Hölder's inequality, we have: $(q = r/p > 1)$

$$\int |f|^p d\sigma = \int |f|^p \cdot 1 d\sigma \le (\int (|f|^p)^q d\sigma)^{1/q} (\int 1^{q'} d\sigma)^{1/q'} = (\int |f|^r d\sigma)^{p/r} < \infty,$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Definition 1.3. *If $f \in L^1(T)$, define Fourier coefficients of f as follows: for $n = 0, \pm 1, \pm 2, \cdots$,*

$$a_n(f) = \int f \chi^{-n} d\sigma = \frac{1}{2\pi} \int_{\pi}^{\pi} F(x) e^{-inx} dx.$$

We now formally introduce the series

$$f(e^{ix}) \sim \sum_{n=-\infty}^{\infty} a_n(f) e^{inx}.$$

The series is called the Fourier series of f. Whenever we speak of convergence or summability of a Fourier series, we are always concerned with the limit, ordinary or generalized, of the symmetric partial sums.

Theorem 1.2. *Let f be a function on T defined as $f = \sum_{-\infty}^{\infty} a_n \chi^n$ so that the right-hand side series converges uniformly. Then $a_n(f) = a_n$ for all n. That is, the Fourier series of f is the series $\sum_{-\infty}^{\infty} a_n \chi^n$.*

Proof: To compute $a_n(f)$, we integrate $\int f \chi^{-n} = \int (\sum_k a_k \chi^k) \chi^{-n}$. To integrate the latter, we integrate term-by-term. It is worth noting that if $\sum_{-\infty}^{\infty} a_n \chi^n$ converges uniformly on T for some ordering of the series, then $a_n(f) = a_n$ and the series is the Fourier series of f. Assume that the series converges uniformly to $g(x)$ on T for some other ordering. Then $a_n(g) = a_n$ so that $a_n(f) = a_n(g)$ and $f = g, a.e.$ Since f and g are continuous, $f = g$ everywhere on T.

Theorem 1.3. *(1)*

$$|a_n(f)| \le \|f\|_1, \ \forall n$$

(2) A more precise result: (Bessel's inequality) If $f \in L^2(T)$, then

$$\sum |a_n(f)|^2 \le \|f\|_2^2.$$

Proof: (1) is trivial. As to (2), let

$$f(e^{ix}) \sim \sum_{n=-\infty}^{\infty} a_n(f)e^{inx}$$

be the Fourier series of f. Define the symmetric partial sums as

$$f_N = \sum_{n=-N}^{N} a_n(f)\chi^n.$$

We can obtain (2) directly from the following computation:

$$\begin{aligned}
\|f - f_N\|_2^2 &= \|f\|_2^2 + \|f_N\|_2^2 - 2Re\int f\overline{f_N}d\sigma \\
&= \|f\|_2^2 + \sum_{-N}^{N} |a_n(f)|^2 - 2\sum_{-N}^{N} Re(\overline{a_n(f)}\int f\overline{\chi}^n d\sigma) \\
&= \|f\|_2^2 + \sum_{-N}^{N} |a_n(f)|^2 - 2\sum_{-N}^{N} Re(\overline{a_n(f)}\int f\chi^{-n} d\sigma) \\
&= \|f\|_2^2 + \sum_{-N}^{N} |a_n(f)|^2 - 2\sum_{-N}^{N} Re(\overline{a_n(f)}a_n(f)) \\
&= \|f\|_2^2 - \sum_{-N}^{N} |a_n(f)|^2.
\end{aligned}$$

Thus,

$$\|f - f_N\|_2^2 = \|f\|_2^2 - \sum_{-N}^{N} |a_n(f)|^2$$

so that

$$\|f\|_2^2 - \sum_{-N}^{N} |a_n(f)|^2 \geq 0, \quad \forall N.$$

We have (2) by taking $N \to \infty$.

Corollary 1.1. *If $f \in L^2(T)$, then the Fourier partial sums $f_N \to f$ in $L^2(T)$ iff*

$$\|f\|_2^2 = \sum |a_n(f)|^2.$$

This equality is called the Parseval relation.

Proof: The corollary follows directly from

$$\|f - f_N\|_2^2 = \|f\|_2^2 - \sum_{-N}^{N} |a_n(f)|^2.$$

Theorem 1.4. *(Riesz-Fischer Theorem) Let* $\{a_n\} \in l^2$*. Then there is* $f \in L^2(T)$ *so that* $a_n(f) = a_n$ *for all* n *and* $\|f\|_2^2 = \sum |a_n|^2$.

Proof: Define $f_N = \sum_{-N}^{N} a_n \chi^n$. Then f_N is Cauchy sequence in $L^2(T)$. Let f be the limit of f_N in $L^2(T)$.

We verify that $a_n(f) = a_n$ for all n. Fix n and let $N \geq n$. Note that $a_n = \int f_N \chi^{-n}$. Then

$$|a_n(f) - a_n| = |\int f\chi^{-n} - \int f_N\chi^{-n}| \leq \|f - f_N\|_2 \to 0, \quad \text{as } N \to \infty.$$

To prove $\|f\|_2^2 = \sum |a_n|^2$ we note that $f_N \to f$ in $L^2(T)$ and so $\|f_N\|_2 \to \|f\|_2$. Since $\|f_N\|_2^2 = \sum_{-N}^{N} |a_n|^2 \to \sum |a_n|^2$, $\|f\|_2^2 = \sum |a_n|^2$.

Theorem 1.5. *For every* $f \in L^2(T)$*, the Parseval relation*

$$\|f\|_2^2 = \sum |a_n(f)|^2$$

holds. Equivalently, for every $f \in L^2(T)$*,* $f_N \to f$ *in* $L^2(T)$ *as* $N \to \infty$.

Proof: Clearly, the Parseval relation holds for all trigonometric polynomials (because the Fourier series of any trigonometric polynomial is itself). Therefore, $\mathcal{F}$ (Fourier transform that carries $f \in L^2$ to $\{a_n(f)\} \in l^2$) is an isometry from trigonometric polynomials in the norm of $L^2(T)$ into l^2, and its range consists of all sequence $\{a_n\}$ such that $a_n = 0$ from some n on.

Note that the range is a dense subset of l^2. If we prove that the family of trigonometric polynomials is dense in $L^2(T)$, then $\mathcal{F}$ has a unique continuous extension, also denoted as $\mathcal{F}$, to a linear isometry of all of $L^2(T)$ onto l^2. This extension $\mathcal{F}$ must be the Fourier transform. (To show this, one needs to prove that for any $f \in L^2(T)$, $\mathcal{F}(f) = \{a_n(f)\}$.) Therefore, the Fourier transform is an isometry from $L^2(T)$ onto l^2.

Theorem 1.6. $\{\chi^n\}$ *is complete. More precisely, if all the Fourier coefficients of* $f \in L^1$ *are zero, then* $f = 0$ *a.e.*

Proof: See ([1])

Theorem 1.7. *(Mercer's Theorem) For any* $f \in L^1(T)$*,* $a_n(f) \to 0$ *as* $n \to \pm\infty$.

Proof: If $f \in L^2(T)$ then the theorem follows from the Bessel's inequality. If $f \in L^1(T)$, choose $f_k \in L^2(T)$ so that $f_k \to f$ in $L^1(T)$. Then for each n, $|a_n(f_k) - a_n(f)| = |a_n(f_k - f)| \leq \|f_k - f\|_1 \to 0$ as $k \to \infty$. This shows that $\{a_n(f_k)\}$ converges to $\{a_n(f)\}$ as $k \to \infty$ uniformly in n. Now, write $|a_n(f)| \leq |a_n(f) - a_n(f_k)| + |a_n(f_k)|$ and for any $\epsilon > 0$, choose k so that $|a_n(f) - a_n(f_k)| < \epsilon$ for all n. By fixing this k and letting n large enough, we get $|a_n(f_k)| < \epsilon$.

Theorem 1.8. *If $f \in L^1(T)$ and $\dfrac{f(e^{ix})}{x}$ is integrable on $(-\pi, \pi)$, then $\sum_{n=-M}^{N} a_n(f) \to 0$ as $M, N \to \infty$ (independently).*

Proof: By the hypothesis,

$$g(e^{ix}) = \frac{f(e^{2ix})}{sinx} \in L^1(T).$$

(Note: the behavior of g near $\pm\pi$ is analogous to that of $\dfrac{f(e^{ix})}{x}$ near 0). Rewriting

$$f(e^{2ix}) = \frac{(e^{ix} - e^{-ix})g(e^{ix})}{2i}$$

and integrating against $\chi^{-2n} d\sigma$, we get

$$2ia_n(f) = a_{2n-1}(g) - a_{2n+1}(g), \quad \forall n.$$

Hence, (telescoping sum), as $M, N \to \infty$,

$$2i \sum_{-M}^{N} a_n(f) = a_{-2M-1} - a_{2N+1} \to 0.$$

(It is worth noting that the gist of the proof is considering $f(e^{2ix})$ and ending up with a telescoping sum.)

Corollary 1.2. *If $f \in L^1$ and f satisfies Lipschitz condition at e^{it}, then the Fourier series of f converges to f at that point. That is, $\sum a_n(f)e^{int} \to f(e^{it})$.*

Proof: Without loss of generality, we may assume that $t = 0$ and $f(1) = 0$ and show that $\sum a_n(f) \to 0$.

Assume that f satisfies the Lipschitz condition at e^{it}, that is, there is a neighborhood of t so that for any x in that neighborhood, $|f(e^{ix}) - f(e^{it})| \leq K|x - t|^\alpha$ for some $0 < \alpha \leq 1$. In our case of $t = 0$ and $f(1) = 0$, this means that $|f(e^{ix})| \leq K|x|^\alpha$, for x close to 0. Therefore, $\dfrac{f(e^{ix})}{x}$ is integrable on $(-\pi, \pi)$. Now the corollary follows from the above theorem.

Theorem 1.9. *(Principle of localization) If $f, g \in L^1(T)$ and $f = g$ on some interval, then at each interior point of the interval, their Fourier series are equi-convergent.*

Proof: Note that

$$\sum_{-M}^{N} a_n(f)\chi^n - \sum_{-M}^{N} a_n(g)\chi^n = \sum_{-M}^{N} a_n(f - g)\chi^n.$$

Since $f - g = 0$ on some interval, $f - g$ satisfies Lipschitz condition at each interior point of that interval. Therefore,

$$\sum_{-M}^{N} a_n(f - g)\chi^n \to 0.$$

This completes the proof.

Theorem 1.10. *Suppose $f(e^{ix}) \in L^1(T)$ and $\dfrac{f(e^{ix}) + f(e^{-ix})}{x}$ is integrable on $(-\pi, \pi)$. Show that*

$$\sum_{-N}^{N} a_n(f) \to 0 \, as \, N \to \infty.$$

Proof: Let $g(e^{ix})$ be such that

$$f(e^{2ix}) - f(e^{-2ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})g(e^{ix}).$$

Note that g is integrable on $(-\pi, \pi)$ by the hypothesis. Integrating against $\chi^{-2n}d\sigma$ we have

$$2i(a_n(f) - a_{-n}(f)) = a_{2n-1}(g) - a_{2n+1}(g).$$

Adding up these equalities for $n = 0, \pm 1, \cdots, \pm N$ we have

$$2i\sum_{-N}^{N}(a_n(f) + a_{-n}(f)) = 4i\sum_{-N}^{N} a_n(f) = a_{-2N-1}(g) - a_{2N+1}(g) \to 0, \quad \text{as} \quad N \to \infty.$$

It is worth noting that, under the hypothesis of the theorem, it is not necessarily true that $\sum_{-M}^{N} a_n(f) \to 0$ as $N, M \to \infty$ independently. For example, let $f(e^{ix}) = -1$ on $(-\pi, 0)$ and $= 1$ on $(0, \pi)$. Then $a_n(f) = \frac{-i}{n\pi}$ for n odd, $= 0$ for n even and

$$f(e^{ix}) \sim \sum \frac{1}{\pi i}(\frac{1 - (-1)^n}{n}).$$

Clearly, $\sum_{-N}^{N} a_n(f) = 0$ for all N. However, if $M = 2N$ and $N = 2k$ then,

$$\sum_{-N}^{M} a_n(f) = \frac{2}{\pi}\sum_{k}^{2k-1}\frac{1}{2n+1} > \frac{2}{\pi}(k\frac{1}{4k-1}) \nrightarrow 0$$

as $k \to \infty$. Note that f (with the definition $f(1) = 0$ is the value of the midpoint in the gap at $x = 0$) satisfies the hypothesis in Theorem 1.10 but not the condition in Theorem 1.8.

Corollary 1.3. *Assume that $f \in L^1$ and f satisfies symmetric Lipschitz condition at e^{it}, that is, there is a neighborhood of t so that for any x in that neighborhood,*

$$|f(e^{ix}) + f(e^{-ix}) - 2f(e^{it})| \le K|x - t|^{\alpha}$$

for some $0 < \alpha \le 1$. Then the Fourier series converges to f at that point. That is, $\sum_{N}^{N} a_n(f)e^{int} \to f(e^{it})$ as $N \to \infty$.

Proof: See ([2])

Theorem 1.11. *For* $0 < r < 1$, $f(e^{ix}) = \dfrac{1}{1 - re^{ix}} = \sum r^n e^{inx}$ *so that* $a_n(f) = r^n$.

2. Convolution

In this section, for any function f defined on T, we write the value of f at e^{ix} as $f(x)$. That is, we always read $f(x)$ as the value of f at e^{ix}.

Convolution on the group T

Definition 2.1. *For* $f, g \in L^1(T)$,

$$f * g(x) = \int f(t)g(x-t)d\sigma(t) = \int f(x-t)g(t)d\sigma(t).$$

Theorem 2.1 (Continuity in $L^p(T)$)**.** *If* $f \in L^p(T)$, $1 \le p < \infty$, *then*

$$\lim_{|h| \to 0} \|f(x+h) - f(x)\|_p = 0.$$

Proof: Let

$$C_p = \{f \in L^p(T) : \|f(x+h) - f(x)\|_p \to 0, \quad \text{as} \quad h \to 0\}.$$

Claim:

(1) A finite linear combination of functions in C_p is in C_p.
Proof: Let f and g be in C_p and a, b are two numbers. Then $\|(af(x+h) + bg(x+h)) - (af(x) + bg(x))\|_p \le |a|\|f(x+h) - f(x)\|_p + |b|\|g(x+h) - g(x)\|_p \to 0$ as $h \to 0$.

(2) If $f_k \in C_p$ and $f_k \to f$ in L^p, then $f \in C_p$.
Proof: Note that $\|f(x+h) - f(x)\|_p \le \|f(x+h) - f_k(x+h)\|_p + \|f_k(x+h) - f_k(x)\|_p + \|f_k(x) - f(x)\|_p = \|f_k(x+h) - f_k(x)\|_p + 2\|f(x) - f_k(x)\|_p$. Since $f_k \in C_p$, we have $\limsup_{|h| \to 0} \|f(x+h) - f(x)\|_p \le 2\|f_k(x) - f(x)\|_p$, and this goes to zero by letting $k \to \infty$.

Clearly, the characteristic function of an interval belongs to C_p. Since the linear combinations of characteristic functions (step functions) of intervals are dense in $L^p(T)$, by using the method of successively approximating more and more general functions, it follows that $L^p(T)$ is contained in C_p.

Theorem 2.2. *(1) If $f \in L^p(T), 1 \leq p \leq \infty$, and $g \in L^{p'}(T)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, then $f * g(x)$ exists everywhere and is continuous with*

$$\|f * g\|_C \leq \|f\|_p \cdot \|g\|_{p'}.$$

(2) If $f \in L^p, 1 \leq p < \infty$, and $g \in L^1(T)$, then $f * g(x)$ exists a.e. as an absolutely convergent integral, $f * g \in L^p(T)$, and $\|f * g\|_{\leq}\|f\|_p \cdot \|g\|_1$.

(3) If $f \in C(T)$ and $g \in L^1(T)$, then $f * g(x)$ exists a.e. as an absolutely convergent integral, $f * g \in C(T)$, and $\|f * g\|_C \leq \|f\|_C \cdot \|g\|_1$.

Proof:

(1) $f \in L^p, 1 \leq p < \infty$ and $g \in L^{p'}$.
Proof: Using the Hölder inequality and continuity in L^p, we see that

$$\begin{aligned} |f * g(x+h) - f * g(x)| &\leq \int |f(x+h-t) - f(x-t)||g(t)|d\sigma(t) \\ &\leq \|f(x+h-\cdot) - f(x-\cdot)\|_p \cdot \|g\|_{p'} \\ &\leq \|f(\cdot + h) - f(\cdot)\|_p \cdot \|g\|_{p'} \end{aligned}$$

approaches 0 as $h \to 0$. The last inequality above holds because the measure σ is translation invariant. The norm estimate follows from the Hölder inequality again. If $p = \infty$, the roles of f and g may be interchanged.

(2) $f \in L^p(T), 1 \leq p < \infty$, and $g \in L^1$.
Proof: Since for almost all u

$$\int |f(x-u)|^p |g(u)| d\sigma(x) = |g(u)| \cdot \|f\|_p^p$$

which belongs to $L^1(T)$, it follows that

$$\int \int |f(x-u)|^p |g(u)| d\sigma(x) d\sigma(u) = \|f\|_p^p \|g\|_1$$

exists as a finite number. Therefore by Fubini's theorem

$$\int \Big(\int |f(x-u)|^p |g(u)| d\sigma(u) \Big) d\sigma(x)$$

exists as well and is equal to $\|f\|_p^p \|g\|_1$. This implies that

$$\int |f(x-u)|^p |g(u)| d\sigma(u)$$

exists for almost every $x \in R$ and belongs to $L^1(T)$. This proves the assertion for $p = 1$.

For $1 < p < \infty$, Hölder's inequality delivers

$$|(f * g)(x)| \leq \left(\int |f(x-u)|^p |g(u)| d\sigma(u) \right)^{1/p} \left(\int |g(u)| d\sigma(u) \right)^{1/p'}.$$

This shows that $f * g(x)$ exists a.e. as an absolutely convergent integral. Moreover,

$$\begin{aligned} \|f * g\|_p &\leq \|g\|_1^{1/p'} \cdot \left(\int \int |f(x-u)|^p |g(u)| d\sigma(u) d\sigma(x) \right)^{1/p} \\ &= \|g\|_1^{1/p'} \|f\|_p \|g\|_1^{1/p} = \|f\|_p \|g\|_1 \end{aligned}$$

(3). Finally, if $f \in C(T)$, it follows as in the proof of (1) that $f * g(x)$ exists for every x, belongs to $C(T)$ and satisfies $\|f * g\|_C \leq \|f\|_C \|g\|_1$.

Definition 2.2. *A (complex) linear algebra is a (complex) linear space $\mathcal{A}$ in which a product is defined such that, for all $x, y, z \in \mathcal{A}, a, b \in C$ $x(yz) = (xy)z$ (associative), $(ax)y = x(ay) = a(xy)$ (commutative with scalar), $x(ay + bz) = a(xy) + b(xz)$ (distributive over addition, or say linear).*

If a linear algebra $\mathcal{A}$ is equipped with a norm under which it is a Banach space, $\mathcal{A}$ is a Banach algebra if $\|xy\| \leq \|x\| \cdot \|y\|$ for all $x, y \in \mathcal{A}$. If furthermore $\mathcal{A}$ has an identity for multiplication, $e = ex = xe$ for all $x \in \mathcal{A}$, then $\|e\| = 1$.

Theorem 2.3. *$L^1(T)$ is a commutative Banach algebra under convolution. This algebra has no identity.*

Proof: We show that this algebra has no identity. If there were such function $\phi \in L^1(T)$ it would mean that $(\phi * f)(x) = f(x)$ for all $f \in L^1(T)$ and almost every $x \in R$. Looking at $\phi * f(x) = \int \phi(x-t) f(t) d\sigma(t)$, since the value of f at x (i.e. e^{ix}) does not depend on the values f takes at other points, we see that $\phi(x-t) = 0$ for almost all $t \neq x$. Otherwise, changing f at t for t in a set of positive measure would change the value of $f(x)$. So $\phi = 0$ a.e., but $\phi * f$ is not identically 0 if f is not identically zero.

Convolution on the line group R

Definition 2.3. *Let f, g be two functions defined and measurable on R. The expression*

$$f * g(x) = \int f(t) g(x-t) d\sigma(t) = \int f(x-t) g(t) d\sigma(t)$$

is called the convolution of f and g.

Theorem 2.4. *(1) Let $f \in L^p, 1 \leq p \leq \infty$, and $g \in L^{p'}$. Then $f * g(x)$ exists everywhere, belongs to C, and $\|f * g\|_C \leq \|f\|_p \|g\|_{p'}$. Moreover, if $1 < p < \infty$, then $f * g \in C_0$, i.e. $f * g \in C$ and $\lim_{|x| \to \infty} f * g(x) = 0$. The same is true for $p = 1$, if, in addition, $g \in C_0$.*

(2) *If $f \in L^p, 1 \leq p < \infty$, and $g \in L^1$, then $f * g(x)$ exists a.e. as an absolutely convergent integral, $f * g \in L^p$, and $\|f * g\|_p \leq \|f\|_p \cdot \|g\|_1$.*

(3) *If $f \in C$ and $g \in L^1$, then $f * g(x)$ exists a.e. as an absolutely convergent integral, $f * g \in C$, and $\|f * g\|_C \leq \|f\|_C \cdot \|g\|_1$.*

Proof: (See [2])

Definition 2.4. *An approximate identity on T is a sequence of functions e_n with these properties:*

(1) *each e_n is nonnegative;*
(2) $\displaystyle\int e_n(t)d\sigma(t) = 1$;
(3) *for every* $0 < \delta < \pi$,

$$\lim_{n\to\infty} \int_{|t|>\delta} e_n(t)d\sigma(t) = 0.$$

Theorem 2.5. *Let X and Y be Banach spaces, and T_n a sequence of linear operators from X to Y with $\|T_n\| \leq K$ for all n, and let D be a dense subset of X. Suppose for each $x \in D$, $T_n x$ converges. Then $T_n x$ converges for all $x \in X$ and if we define $T : X \to Y$ by $Tx = limT_n x$ for all $x \in X$, then T is a linear operator with $\|T\| \leq K$.*

Proof: Let $x \in X$ and $x_k \in D$ with $x_k \to x$ in X. Then $\|T_n x - T_m x\| \leq \|T_n x - T_n x_k\| + \|T_n x_k - T_m x_k\| + \|T_m x_k - T_m x\| \leq 2K\|x - x_k\| + \|T_n x_k - T_m x_k\|$. Given $\epsilon > 0$, find k so that $2K\|x - x_k\| < \epsilon$ and for this fixed k, find N so that if $n, m > N$, then $\|T_n x_k - T_m x_k\| < \epsilon$. So $T_n x$ is a Cauchy sequence in Y. Since Y is Banach, $T_n x$ converges.

Let T be defined as in the theorem. Then T is linear operator and for any x with $\|x\| \leq 1$, $\|Tx\| \leq K$. Therefore, $\|T\| = \sup_{\|x\|\leq 1} \|Tx\| \leq K$.

Theorem 2.6 (Fejér's)**.** *Let $f \in L^p(T)$ with $1 \leq p < \infty$. Then for any approximate identity e_n, $e_n * f \to f$ in $L^p(T)$. If $f \in C(T)$, then $e_n * f \to f$ uniformly.*

Proof: If $f \in C(T)$, then

$$f(x) - e_n * f(x) = \int (f(x) - f(x - t))e_n(t)d\sigma(t).$$

Note that f is uniformly continuous on T. Given $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(x - t)| < \epsilon$ for $|t| \leq \delta$. Denote by I the part of the integral over $|t| < \delta$, and by J the integral over the complementary interval. Clearly,

$$|I| \leq \epsilon \int_{|t|\leq\delta} e_n(t)d\sigma(t) \leq \epsilon$$

and

$$|J| \leq 2M \int_{|t|>\delta} e_n(t)d\sigma(t) \to 0$$

as $n \to \infty$.

Now let $f \in L^p$ and $1 \leq p < \infty$. For continuous f, the uniform convergence of $e_n * f$ to f implies convergence in L^p. Let $T_n(f) = e_n * f$. Then T_n a linear operator from $L^p(T) \to L^p(T)$ such that $\|T_n(f)\|_p \leq \|e_n\|_1\|f\|_p$, i.e., $\|T_n\| \leq 1$. Note that $T_n(f)$ converges in $L^p(T)$ for every $f \in C(T)$ and $C(T)$ is dense in $L^p(T)$. Therefore, Theorem 1.4 asserts that $T_n(f)$ converges in $L^p(T)$ for every $f \in L^p(T)$ and if we define $T(f) = L^p - \lim T_n(f)$ then T is a linear operator on L^p with bound ≤ 1. We prove that T is an identity on L^p. In fact, $T(f) = f$ for all $f \in C(T)$ and $C(T)$ is dense in $L^p(T)$. Let $f \in L^p$ and let $f_k \in C(T)$ with $f_k \to f$ in L^p. Then $T(f) = \lim T(f_k) = \lim f_k = f$ for all f, that is, $L^p - \lim e_n * f = f$ for all $f \in L^p$.

Theorem 2.7. *Let e_n be an approximate identity on T, and $g \in L^\infty(T)$. Then $e_n * g \to g$ in the weak* topology of L^∞. That is, for every $f \in L^1(T)$,*

$$\lim_{n\to\infty} \int \Big((e_n * g)(x) - g(x)\Big) f(x)d\sigma(x) \to 0.$$

Proof: If $g \in L^\infty$ then

$$e_n * g(x) - g(x) = \int (g(x-u) - g(x))e_n(u)d\sigma(u).$$

By Fubini's theorem, for every $f \in L^1(T)$, and $0 < \delta < \pi$,

$$\begin{aligned}
. \quad & \int \Big((e_n * g)(x) - g(x)\Big) f(x)d\sigma(x) \\
= \quad & \int \Big(\int (g(x-u) - g(x))e_n(u)d\sigma(u)\Big) f(x)d\sigma(x) \\
= \quad & \int \int (g(x-u)f(x) - g(x)f(x))d\sigma(x)e_n(u)d\sigma(u) \\
= \quad & \int \int (g(x)f(x+u) - g(x)f(x))d\sigma(x)e_n(u)d\sigma(u) \\
= \quad & \int \int (f(x+u) - f(x))g(x)d\sigma(x)e_n(u)d\sigma(u).
\end{aligned}$$

But since $f \in L^1(T)$ is continuous in mean, for each $\epsilon > 0$, there is a $\delta > 0$ such that $\|f(\cdot + u) - f(\cdot)\|_1 < \epsilon$ for all $|u| \leq \delta$. Denote by I the part of the integral over $|u| < \delta$, and by J the integral over the complementary interval. Then

$$|I| \leq sup_{|u|\leq\delta} \Big| \int g(x)(f(x+u) - f(x))d\sigma(x)\Big| \leq \|g\|_\infty \epsilon.$$

Furthermore,

$$|J| \leq 2\|g\|_\infty \|f\|_1 \int_{\delta \leq |u| \leq \pi} |e_n(u)| d\sigma(u).$$

It follows $J \to 0$ as $n \to \infty$.

Theorem 2.8. *Let e_k be an approximate identity on T. Then for each n, $a_n(e_k) \to 1$ as $k \to \infty$.*

Proof: For each n, $e_k * e^{in\cdot}(e^{ix}) \to e^{inx}$ uniformly. In particular, let $x = 0$, $e_k * e^{in\cdot}(1) \to 1$. That is, $a_n(e_k) \to 1$.

Theorem 2.9. *Assume that there is an approximate identity e_k on T consisting of trigonometric polynomials. Then trigonometric polynomials are dense in $L^p(T)$, $1 \leq p < \infty$.*

Proof: Let $f \in L^p$, $1 \leq p < \infty$. Then $f \in L^1(T)$ and $a_n(e_k * f) = a_n(e_k)a_n(f)$. Since e_k is a trigonometric polynomial, for each k, $a_n(e_k) = 0$ so that $a_n(e_k * f) = 0$ for large n. Thus, $e_k * f$ is a trigonometric polynomial. Since $e_k * f \to f$ in $L^p(T)$, $1 \leq p < \infty$, trigonometric polynomials are dense in $L^p(T)$.

Theorem 2.10 (Converse of Hölder's Inequality)**.** *Let $1 \leq p < \infty$. If $\int |fg| d\sigma \leq k$ for every $g \in L^{p'}(T)$ with norm 1, then $f \in L^p(T)$ and $\|f\|_p \leq k$.*

Proof: Let $f_n = f$, if $x \in \{x : |f(x)| \leq n\}$; $f_n = 0$ if x is in the complement. Then $f_n \in L^p(T)$ for any p and $|f_n| \uparrow |f|$ a.e. Let

$$g = \frac{|f_n|^{p-1}}{\|f_n\|_p^{p-1}}.$$

Then $\|g\|_{p'} = 1$. By assumption, we have

$$\frac{1}{\|f_n\|_p^{p-1}} \int |f||f_n|^{p-1} d\sigma = \frac{1}{\|f_n\|_p^{p-1}} \int |f_n|^p d\sigma \leq k.$$

By B. Levi's theorem (Monotone Convergence Theorem), $\|f\|_p \leq k$. The proof can be extended to $L^p(R)$ by defining $f_n = f$ for $x \in \{x \in R : |x| \leq n \quad \text{and} \quad |f(x)| \leq n\}$.

3. Unicity Theorem, Parseval Relation

Theorem 3.1 (Unicity Theorem)**.** *If $f \in L^1(T)$ and $a_n(f) = 0$ for all n, then $f = 0$ a.e.*

Proof: If f has continuous derivative then f satisfies the Lipschitz condition at every point so that $\sum a_n(f)e^{inx}$ converges to $f(x)$ everywhere. If $a_n(f) = 0$ for all n, then the sum is zero so that $f = 0$ everywhere.

Let e_n be an approximate identity in $L^1(T)$ so that e_n's are continuously differentiable. Then for every $f \in L^1$, $e_n * f$ is continuously differentiable. Let f be such that $a_n(f) = 0$ for all n. Then for every n, $a_k(e_n * f) = a_n(e_k)a_n(f) = 0$ for all k. By the first part of proof, $e_n * f = 0$ everywhere. Thus, f, as a limit of $e_n * f$ in L^1, is zero almost everywhere.

Corollary 3.1. *The trigonometric polynomials are dense in $L^p(T), 1 \leq p < \infty$.*

Proof: If the trigonometric polynomials are not dense in $L^p(T)$, then there exists a nonzero linear functional l on $L^p(T)$ so that $l(h) = 0$ for all h in the closure of the trigonometric polynomials. It follows that there exists $g \in L^{p'}, 1 < p' \leq \infty, g \neq 0, a.e.$ so that $l(h) = \int g\overline{h}d\sigma = 0$ for all trigonometric polynomials h. This means $a_n(g) = 0$ for all n so that $g = 0$ a.e., contrary to hypothesis.

Theorem 3.2 (Parseval Relation)**.** *The Parseval relation*

$$||f||_2^2 = \sum |a_n(f)|^2$$

holds for every $f \in L^2(T)$.

Proof: The Parseval relation holds for all trigonometric polynomials. It is also valid for every function that is the limit of trigonometric polynomials in $L^2(T)$. Since every function in $L^2(T)$ is such a limit, the Parseval relation holds for every $f \in L^2(T)$.

Let $C_0(X)$ denote the subset of $C(X)$ consisting of all $f \in C(X)$ such that for every $\epsilon > 0$, there is a compact subset F of X such that $|f(x)| < \epsilon$ for all $x \in F' \bigcap X$. If X is compact then $C_0(X) = C(X)$.

Theorem 3.3 (F. Riesz Representation Theorem)**.** *Let X be a locally compact Hausdorff space. Then the mapping $T(\mu) = l_\mu$ is a norm-preserving linear mapping of $M(X)$ onto $C_0(X)^*$, where $l_\mu(f) = \int_X f d\mu$, $f \in C_0(X)$. Thus $M(X)$ is a Banach space, and $M(X)$ and $C_0(X)^*$ are isomorphic as Banach spaces.*

Proof: See ([3])

Theorem 3.4 (Weierstrass)**.** *The trigonometric polynomials are dense in $C(T)$.*

Proof: We want to show that any continuous linear functional l on $C(T)$ that vanishes on all trigonometric polynomials is null. Note that such a linear functional can be realized by a bounded, complex Borel measure μ on T (realized as $[-\pi, \pi)$) in the way that

$$l(h) = \int h(e^{-ix})d\mu(x), \qquad h \in C(T).$$

Thus, the proof of Weierstrass' theorem is converted to proving that if $\int e^{-inx}d\mu(x) = 0$ for all n then μ is null. We prove this fact in the following theorem, which is a stronger statement of Unicity theorem when we identify a function with the measure $f(e^{ix})d\sigma(x)$.

Theorem 3.5 (Unicity Theorem)**.** *Let μ be Borel measure on T. The Fourier-Stieltjes coefficients of μ are defined to be*

$$\hat{\mu}(n) = a_n(\mu) = \int e^{-inx} d\mu(x).$$

If $\hat{\mu}(n) = 0$ for all n, then μ is null.

Proof: Note that if $f \in C(T)$ and $a_n(f) = 0$ for all n then $f = 0$. This is the weakest Unicity theorem of all. We use this version and the Riesz theorem to prove the strongest version as stated in the theorem.

For $g \in C(T)$, define the convolution

$$\mu * g(e^{ix}) = \int g(e^{i(x-t)}) d\mu(t).$$

We show that it is a continuous function. As usual, we write $g(e^{ix})$ as $g(x)$ for simplicity. Note that

$$\begin{aligned} |\mu * g(x+h) - \mu * g(x)| &= |\int (g(x+h-t) - g(x-t)) d\mu(t)| \\ &\leq \|g(\cdot + h) - g(\cdot)\|_C \int d|\mu(t)| \\ &= \|g(\cdot + h) - g(\cdot)\|_C \|\mu\|. \end{aligned}$$

Since $|\mu|(T)$ is finite and g is uniformly continuous, the last expression tends to zero as $h \to 0$.

The Fourier coefficients of $\mu * g$ are $a_n(\mu * g) = a_n(\mu) a_n(g)$. By assumption, $a_n(\mu) = 0$ for all n. Thus, $a_n(\mu * g) = 0$ for all n and, since $\mu * g \in C(T)$, $\mu * g(x) = 0$ at every x. In particular, $\mu * g(1) = 0$. Since g was an arbitrary function in $C(T)$, $(\mu * g)(1)$ is the null functional on $g \in C(T)$, and so by the Riesz theorem, μ is null measure. Note that $\mu * g(1) = \int g(e^{-ix}) d\mu(x)$. The theorem is proved.

Definition 3.1. *The convolution of two Borel measures on T, say μ and ν, is defined to be the measure $\mu * \nu$ such that*

$$(\mu * \nu) * h(1) = (\mu * (\nu * h))(1).$$

Clearly, by translating h we see that $(\mu * \nu) * h(e^{ix}) = (\mu * (\nu * h))(e^{ix})$, for $h \in C(T)$ and all x. That is, for all x,

$$\int h(e^{i(x-t)}) d(\mu * \nu)(t) == \int \int h(e^{i(x-s-t)}) d\nu(t) d\mu(s).$$

Letting $x = 0$ gives the equality in the above definition of convolution of two measures.

Theorem 3.6. $\mu * \nu \in M(T)$.

Proof: Define the functional $l(h) = (\mu * (\nu * h))(1)$ on $C(T)$. Clearly, it is linear. Observe that $\|\nu * h\|_\infty \leq \|\nu\|\|h\|_\infty$ for any measure $\nu \in M(T)$ and $h \in C(T)$. Applying twice, we have $|l(h)| \leq \|\mu\|\|\nu\|\|h\|_\infty$. Therefore, l is a continuous linear functional on $C(T)$. By the Riesz theorem, there is $\gamma \in M(T)$ so that $l(h) = \int h(e^{-it})d\gamma(t) = \gamma * h(1)$ for all $h \in C(T)$. Then $\mu * \nu$ is defined to be the measure $\gamma \in M(T)$. Moreover, $\|\mu * \nu\| \leq \|\mu\|\|\nu\|$.

Theorem 3.7. *$M(T)$ is a commutative Banach algebra under the convolution.*

Proof: Observe that, for χ defined as $\chi(e^{ix}) = e^{ix}$, $(\mu * \chi^n)(e^{ix}) = \int e^{in(x-t)}d\mu(t) = a_n(\mu)\chi^n(e^{ix})$. Hence,

$$\begin{aligned} a_n(\mu * \nu) \cdot \chi^n(1) &= ((\mu * \nu) * \chi^n)(1) = (\mu * (\nu * \chi^n))(1) \\ &= a_n(\nu)(\mu * \chi^n)(1) = a_n(\nu)a_n(\mu) \cdot \chi^n(1). \end{aligned}$$

It follows from the Unicity theorem that $\mu * \nu = \nu * \mu *$, and $(\mu * \nu) * \gamma = \mu * (\nu * \gamma)$.

Proposition 3.1. *The convolution of a measure with an integrable function is the same as that induced by considering $L^1(T)$ as a subalgebra of $M(T)$. In other words, $\mu*(fd\sigma) = (\mu*f)d\sigma$, where $f \in L^1(T)$ and is identified with the measure $fd\sigma$, while $\mu * f \in L^1(T)$.*

Solution: We have shown that if $f \in C(T)$, then $\mu * f \in C(T)$. Let $f \in L^p$, $1 \leq p < \infty$. It follows by Fubini's theorem that

$$\int (\int |f(e^{i(x-t)})|^p d|\mu(t)|)d\sigma(x) = \|f\|_p\|\mu\|.$$

This implies that $\int |f(e^{i(x-t)}|^p d|\mu(t)|$ exists for almost all x and belongs to $L^1(T)$. This proves that if $f \in L^1(T)$, then so is $f * \mu$. If $1 < p < \infty$, by Hölder's inequality

$$|f * \mu(e^{ix})| \leq \int |f(e^{i(x-t)})d|\mu(t)| \leq (\int |f(e^{i(x-t)})|^p d|\mu(t)|)^{1/p}(\int d|\mu(t)|)^{1/p'}.$$

Since μ is bounded, $(f * \mu)(e^{ix})$ exists a.e. Further, one can show that $f * \mu \in L^p(T)$ as the proof of the assertion that if $f \in L^p$ and $g \in L^1$, then $f * g \in L^p$.

We verify that both $\mu * (fd\sigma)$ and $(\mu * f)d\sigma$ are in $M(T)$ and, as linear functionals acting on $h \in C(T)$,

$$\int h(e^{-it})d(\mu * fd\sigma)(t) = \int h(e^{-it})(\mu * f)(e^{it})d\sigma(t).$$

Thus $\mu * (fd\sigma) = (\mu * f)d\sigma$.

Note that the right hand side equals, using Fubini's theorem,

$$\begin{aligned} \int h(e^{-it}) \int f(e^{i(t-u)})d\mu(u)d\sigma(t) &= \int \Big(\int h(e^{-it})f(e^{i(t-u)})d\sigma(t)\Big)d\mu(u) \\ &= \int (f * h)(e^{-iu})d\mu(u). \end{aligned}$$

The left hand side, by the definition of the convolution of two measures, equals

$$\begin{aligned}\int h(e^{-it})\int f(e^{i(t-u)})d\mu(u)d\sigma(t) &= \int\int h(e^{-i(t+u)})f(e^{it})d\sigma(t)d\mu(u)\\ &= \int (f*h)(e^{-iu})d\mu(u).\end{aligned}$$

The following theorem shows that the discrete part of a measure μ can be extracted from its Fourier-Stieltjes series:

Theorem 3.8 (Wiener)**.** *If $\mu \in M(T)$ has (its discrete part) point masses a_n, then*

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{-N}^{N}|\hat{\mu}(n)|^2 = \sum |a_n|^2.$$

Consequently, $\mu \in M(T)$ is continuous if and only if

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{-N}^{N}|\hat{\mu}(n)|^2 = 0.$$

Proof: Let $\tilde{\mu}(E) = \overline{\mu}(-E)$ for all Borel sets E of T. Then

$$\begin{aligned}\hat{\tilde{\mu}}(n) &= \int e^{-inx}d\tilde{\mu}(x) = \int e^{inx}d\tilde{\mu}(-x)\\ &= \int e^{inx}d\overline{\mu}(x) = \overline{\int e^{-inx}d\mu(x)} = \overline{\hat{\mu}}\end{aligned}$$

Hence $a_n(\mu*\tilde{\mu}) = a_n(\mu)a_n(\tilde{\mu}) = |a_n(\mu)|^2$ so that

$$\frac{1}{2N+1}\sum_{-N}^{N}|\hat{\mu}(n)|^2 = \int \frac{1}{2N+1}\sum_{-N}^{N}e^{-inx}d(\mu*\tilde{\mu})(x).$$

As $N\to\infty$, the integrand tends boundedly to 1 at $x = 0$, and to 0 elsewhere on $(-\pi,\pi)$. Hence the limit is the mass of $\mu*\tilde{\mu}$ at 0.

To prove $\mu*\tilde{\mu}(\{0\}) = \sum |a_n|^2$, we define

$$h_\epsilon(e^{it}) = \begin{cases} 1 & \text{if } |t| < \epsilon,\\ 0 & \text{otherwise.}\end{cases}$$

Then, using the Bounded Convergence Theorem we can write

$$(\mu*\tilde{\mu})(\{0\}) = \lim_{\epsilon\uparrow 0}\int h_\epsilon(e^{-it})d\mu*\tilde{\mu}(t),$$

which can be viewed as the value of $(\mu * \tilde{\mu}) * h_\epsilon(e^{ix})$ at $x = 0$. By the definition of the convolution of two measures, we can rewrite

$$\begin{aligned}(\mu * \tilde{\mu}) * h_\epsilon(1) &= \mu * (\tilde{\mu} * h_\epsilon)(1) \\ &= \int\int h_\epsilon(e^{i(-s-t)})d\tilde{\mu}(t)d\mu(s) \\ &= \int\int h_\epsilon(e^{i(-s+t)})d\tilde{\mu}(-t)d\mu(s) \\ &= \int\int h_\epsilon(e^{i(t-s)})d\overline{\mu}(t)d\mu(s).\end{aligned}$$

We calculate the last convolution. For each $e^{is} \in T$,

$$\lim_{\epsilon\to 0}\int h_\epsilon(e^{i(t-s)})d\overline{\mu}(t) = \lim_{\epsilon\to 0}\int_A d\overline{\mu}(t) = \lim_{\epsilon\to 0}\overline{\mu}(A) = \overline{\mu}(\{e^{is}\}),$$

where $A = \{e^{it} \in T : s - \epsilon < t < s + \epsilon\}$ shrinks to $\{e^{is}\}$ as $\epsilon \to 0$. Also note that $\mu \in M(T)$ so that $|\mu|(T)$ is finite. It follows that

$$|\int h_\epsilon(e^{i(t-s)})d\overline{\mu}(t)| \leq \int d|\overline{\mu}(t)| = |\mu|(T) < \infty.$$

Therefore, by the Lebesgue Bounded Convergence Theorem,

$$\begin{aligned}\int\int h_\epsilon(e^{i(t-s)})d\overline{\mu}(t)d\mu(s) &\to \int \overline{\mu}(\{s\})d\mu(s) \\ &= \sum \overline{\mu}(\{e^{i\theta_k}\})\mu(\{e^{i\theta_k}\}) = sum_k|a_k|^2,\end{aligned}$$

as $\epsilon \to 0$.

Proposition 3.2. *If μ is a measure and e_n is an approximate identity on T, then $\mu * e_n$ tends to μ in the weak* topology of $M(T)$ as the dual of $C(T)$. That is,*

$$\lim_{n\to\infty}\int h(e^{-ix})\int e_n(e^{i(x-t)})d\mu(t)d\sigma(x) \to \int h(e^{-it})d\mu(t)$$

for all $h \in C(T)$.

Proof: By Fubinis' theorem,

$$\int h(e^{-ix})\int e_n(e^{i(x-t)})d\mu(t)d\sigma(x) = \int\int h(e^{-ix})e_n(e^{i(x-t)})d\sigma(x)d\mu(t).$$

Note that

$$\int h(e^{-ix})e_n(e^{i(x-t)})d\sigma(x) = e_n * h(e^{-it}) \to h(e^{-it})$$

uniformly on T. Taking limit under the outer integral gives rise to the desired limit.

4. The Classical Kernels

The Dirichlet kernel

For each $n = 0, 1, 2, \cdots$, the partial sum $S_n(f)$ of Fourier series of a function f can be written as

$$\begin{aligned} S_n(f)(e^{ix}) = \sum_{-n}^{n} a_k(f)e^{ikx} &= \sum_{-n}^{n} \int f(e^{it})e^{ik(x-t)}d\sigma(t) \\ &= \int f(e^{i(x-t)}) \sum_{-n}^{n} e^{ikt} d\sigma(t) = (f * D_n)(e^{ix}), \end{aligned}$$

where

$$D_n(e^{ix}) = \sum_{-n}^{n} e^{ikx} = \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}, n = 0, 1, 2, \cdots.$$

Since $D_n(e^{-ix}) = D_n(e^{ix})$ (even), the Fourier series $S_n(f)$ of f converges at $x = 0$ as $n \to \infty$ if and only if

$$\begin{aligned} \lim_{n\to\infty}(f * D_n)(1) &= \lim_{n\to\infty} \int f(e^{-it})D_n(e^{it})d\sigma(t) \\ &= \lim_{n\to\infty} \int f(e^{it})D_n(e^{it})d\sigma(t) \end{aligned}$$

exists. If we take $f(e^{ix}) = -1$ on $(-\pi, 0)$ and $= 1$ on $(0, \pi)$, then

$$f \sim \sum_{-\infty}^{\infty} \frac{1}{\pi i}(\frac{1-(-1)^n}{n})e^{inx}.$$

We have $(f * D_n)(1) = 0$ for all n, but $\sum_{-M}^{N} a_n(f)$ does not converge. Hence, the "convergence" of a Fourier series in complex form should be the convergence of symmetric sums.

Theorem 4.1. *If we define $L_n = \|D_n\|_1$ as the Lebesgue constant, then $L_n = \frac{4}{\pi^2}\ln n + O(1)$. Thus D_n do not form an approximate identity.*

Proof: Note

$$\|D_n\|_1 = \int_{-\pi}^{\pi} |D_n| d\sigma = \frac{1}{\pi}\int_0^{\pi} |D_n| dx.$$

Since

$$\int_0^{\pi} |\sin(n+\frac{1}{2})x| |\frac{1}{\sin\frac{x}{2}} - \frac{1}{\frac{x}{2}}| dx$$

is majorized by an absolute constant, we estimate

$$2\int_0^{\pi} |\frac{\sin(n+\frac{1}{2})x}{x}| dx,$$

which can be written as, if let $(n+\frac{1}{2})x = u$,

$$2\int_0^{(n+\frac{1}{2})\pi} \frac{|\sin u|}{u} du.$$

We may disregard the parts of this integral over $(0,\pi)$ and $(n\pi, (n+\frac{1}{2}\pi)$, since the integrand is bounded. In view of the periodicity of $\sin u$, what remains can be written as

$$2\int_\pi^{n\pi} \frac{|\sin u|}{u} du = 2\int_0^\pi \sin u (\sum_{k=1}^{n-1} \frac{1}{u+k\pi}) du.$$

For $0 \le u \le \pi$, the sum is contained between $\frac{1}{\pi}\sum_{k=2}^n \frac{1}{k}$ and $\frac{1}{\pi}\sum_{k=1}^{n-1} \frac{1}{k}$, and so is strictly of order $\frac{1}{\pi}\ln n$. Collecting estimates, we obtain $\|D_n\|_1 = \frac{4}{\pi^2}\ln n + O(1)$.

Theorem 4.2. *There is a continuous function whose Fourier series diverges at a point.*

Proof: Suppose it were true that $S_n(h) = \sum_{-n}^n a_k(h)$ has a limit (i.e. $S_n(h)(e^{ix})$ converges at $x = 0$) for any $h \in C(T)$. Then we would have $|S_n(h)| \le M_h$ for all n, where M_h is a constant. For each n, S_n is a linear functional on $C(T)$, given as $S_n(h) = \int h(e^{ix})D_n(e^{ix})d\sigma(x)$. If $S_n(h) \le M_h$ for all n and for every $h \in C(T)$, then by the Banach-Steinhaus theorem the sequence of norms $\|S_n\|$ is bounded. But this norm is the Lebesgue constant, which tends to ∞ with n. This proves that $S_n(h)$ is unbounded for some h, and thus divergent.

The Fejér kernel

Define

$$K_n(e^{ix}) = \sum_{-n}^{n} (1 - \frac{|k|}{n}) e^{ikx}.$$

Then

$$\begin{aligned}
f * K_n(e^{ix}) &= \frac{1}{n}\int f(e^{i(x-t)}) \sum_{k=-n}^{n} (n-|k|) e^{ikt} d\sigma(t) \\
&= \frac{1}{n}\int f(e^{i(x-t)})(D_{n-1}(e^{it}) + D_{n-2}(e^{it}) + \cdots + D_1(e^{it}) + D_0) d\sigma(t) \\
&= \frac{1}{n}\sum_{j=0}^{n-1} (f * D_j)(e^{ix}) = \frac{1}{n}\sum_{j=0}^{n-1} S_j(f)(e^{ix}).
\end{aligned}$$

One can easily verify, by multiplying out, that

$$\frac{1}{n}|\sum_{k=0}^{n-1} e^{ikx}|^2 = \sum_{k=-n}^{n} (1 - \frac{|k|}{n}) e^{ikx}.$$

When we sum the geometric series and simplify, we find

$$K_n(e^{ix}) = \frac{1}{n}(\frac{\sin\frac{1}{2}nx}{\sin\frac{1}{2}x})^2.$$

Thus the Dirichlet and Fejér kernels are related by the formula

$$K_{2n+1}(e^{ix}) = \frac{1}{2n+1}D_n^2(e^{ix}).$$

Note that K_n is an approximate identity on T. Thus for any $f \in L^1(T)$, $K_n * f(e^{ix}) \to f(e^{ix})$ at every point of of continuity of f, and the convergence is uniform over every closed interval of continuity. In particular, $K_n * f$ tends to f uniformly everywhere if f is continuous everywhere. It holds also that if $f \in L^p$, $1 \le p < \infty$, then $\|K_n * f - f\|_p \to 0$.

The functions K_n are trigonometric polynomials; this fact has interesting consequence.

(1) Since K_n's are infinitely differentiable, any continuous function h is approximated uniformly by the infinitely differentiable functions (in fact, trigonometric polynomials) $K_n * h$.
(2) We also obtain another proof of the Unicity theorem in $L^1(T)$. Suppose that $a_k(f) = 0$ for all k. Then for each n, $a_k(K_n * f) = a_k(K_n)a_n(f) = 0, \forall k$. Thus the trigonometric polynomial $K_n * f \equiv 0$. Since $\|K_n * f - f\|_1 \to 0$, $f = 0$ a.e.

The Poisson kernel

Define, for $0 < r < 1$,

$$P_r(e^{it}) = \sum_{-\infty}^{\infty} r^{|n|}e^{int}.$$

The series converges absolutely, and we can easily obtain that, if $z = re^{i\theta}, 0 \le r < 1$, then

$$P_r(e^{i(\theta-t)}) = Re(\frac{e^{it}+z}{e^{it}-z}) = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}.$$

One can verify that P_r is an approximate identity (with continuous parameter r). Clearly, if $f \in L^1(T)$ then

$$\sum_{-\infty}^{\infty} a_n(f)r^{|n|}e^{in\theta} == (P_r * f)(e^{i\theta}) = \int P_r(e^{i(\theta-t)})f(e^{it})d\sigma(t).$$

Theorem 4.3. *The Poisson integral $(P_r * f)(e^{i\theta})$ provides the harmonic extension of $f \in L^1(T)$ to the interior of the circle. Moreover,*

(1) If $f \in L^p(T)$ with $1 \le p < \infty$, then $\|P_r * f - f\|_p \to 0$ as $r \uparrow 1$.

(2) *Let $f \in L^1(T)$. At every point t where f is the derivative of its integral (hence almost everywhere) $P_r * f(e^{it}) \to f(e^{it})$ as $r \uparrow 1$ (radial limit). Actually, for almost all t, $P_r * f(e^{i\theta}) \to f(e^{it})$ as $re^{i\theta} \to e^{it}$ nontangentially. This result depends on particular properties of the Poisson kernel, and is not true for all approximate identities.*

Proof: We prove that $P_r * f(e^{ix})$ is harmonic in D (open unit disk). If f is real, then $P_r * f$ is the real part of

$$\int \frac{e^{it}+z}{e^{it}-z} f(e^{it})d\sigma(t),$$

which is an analytic function of $z = re^{i\theta}$ in D. Hence $P_r * f(e^{i\theta})$ is harmonic in D. Since linear combinations of harmonic functions are harmonic, $P_r * f(e^{i\theta})$ is a complex harmonic function on D for any $f \in L^1(T)$, the class of all complex, Lebesgue integrable functions on T.

Theorem 4.4. *Suppose $f \in L^1(T)$ and $f \geq 0$. Then f is the boundary function of a nonnegative harmonic function. If f is bounded, it is the boundary function of a harmonic function with the same bounds.*

Proof: Let $F(z) = P_r * f(e^{ix})$. Then $F(z)$ is harmonic in D such that $\lim_{r\uparrow 1} F(re^{i\theta}) = f(e^{i\theta})$ for a.e. θ. Since P_r is nonnegative, $F(z)$ is certainly positive whenever f is nonnegative. If $|f| \leq M$, then $\|P_r * f\|_\infty \leq M\|P_r\|_1 = M$.

Theorem 4.5. *A harmonic function F in D (open disk) is bounded if and only if it is the Possion integral of some bounded function f on T.*

Proof: We need only to show the necessity. Let F be harmonic and bounded in D. Let $r_n \uparrow 1$ and write $f_n(e^{it}) = F(r_n e^{it})$. The sequence f_n is a bounded sequence in $L^\infty(T)$; hence for some sequence $n_j \to \infty$, f_{n_j} converges in the weak-* topology ($L^\infty(T)$ being the dual of $L^1(T)$) to some function $f(e^{it})$.

Let $re^{i\theta} \in D$, then

$$\begin{aligned}\int f(e^{it})P_r(e^{i(\theta-t)})d\sigma(t) &= \lim_{j\to\infty}\int F(r_{n_j}e^{it})P_r(e^{i(\theta-t)})d\sigma(t)\\ &= \lim_{j\to\infty} F(r_{n_j}re^{it}) = F(re^{i\theta}).\end{aligned}$$

Note that in the above derivation we used the fact that for any harmonic function u in D and $0 \leq r < 1$,

$$\int P_r(e^{i(\theta-t)})u(\rho e^{it})d\sigma(t) = u(\rho re^{i\theta}).$$

This can be verified by considering the representation theorem of harmonic functions in disk : If u is real, continuous on $|z| \leq \rho$ and harmonic in $|z| < \rho$, then for $z = \rho_1 e^{i\theta}$, $\rho_1 < \rho$,

$$u(\rho_1 e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} Re[\frac{\rho e^{it} + z}{\rho e^{it} - z}]u(\rho e^{it})dt.$$

Let $\rho_1 = r\rho$ (Note $0 \leq r < 1$). Then

$$u(r\rho e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} Re[\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}]u(\rho e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} P_r(e^{i(\theta-t)})u(\rho e^{it})dt.$$

For complex harmonic, we consider it as a sum of real part and imaginary part.

5. Summability; Metric Theorems

We have shown the following theorems in the last section:

Theorem 5.1. *The Poisson integral $(P_r * f)(e^{i\theta})$ provides the harmonic extension of $f \in L^1(T)$ to the interior of the circle so that*

(1) If $f \in C(T)$, then $P_r * f$ converges to f uniformly as $r \to 1$.
(2) If $f \in L^p(T)$ with $1 \leq p < \infty$, then $\|P_r * f - f\|_p \to 0$ as $r \uparrow 1$.
(3) Let $f \in L^1(T)$. At every point t where f is the derivative of its integral (hence almost everywhere) $P_r * f(e^{i\theta}) \to f(e^{it})$ as $re^{i\theta} \to e^{it}$ nontangentially.

Theorem 5.2. *Let μ be any finite complex measure on T. Then*

$$P_r * \mu(e^{i\theta}) = \int P_r(e^{i(\theta-t)})d\mu(t)$$

is a harmonic function in D and converges to μ in the weak topology of $M(T)$. That is, for any $h \in C(T)$,*

$$\int h(e^{ix})(P_r * \mu)(e^{ix})d\sigma(x) \to \int h(e^{ix})d\mu(x), \quad r \to 1.$$

Proof: $P_r * \mu(e^{i\theta})$ is a continuous function of θ and $a_n(P_r * \mu) = a_n(P_r)a_n(\mu)$. Thus

$$P_r * \mu(e^{i\theta}) = \sum a_n(\mu)r^{|n|}e^{in\theta}.$$

Since $|a_n(\mu)| \leq |\mu|(T)$, the above series converges uniformly in every compact subset of D and so $P_r * \mu(e^{i\theta})$ is harmonic in D.

For any $h \in C(T)$,

$$\int h(e^{ix})\int(P_r(e^{i(x-t)})d\mu(t)d\sigma(x) = \int\int h(e^{ix})P_r(e^{i(x-t)})d\sigma(x)d\mu(t).$$

Note that $P_r * h$ converges uniformly to h. Taking the limit of above equality as $r \uparrow 1$, we complete the proof of the theorem.

Theorem 5.3. *A harmonic function F in D (open disk) is bounded if and only if it is the Possion integral of some bounded function f on T.*

Proof: We need only to show the necessity. Let F be harmonic and bounded in D. Let $r_n \uparrow 1$ and write $f_n(e^{it}) = F(r_n e^{it})$. The sequence f_n is a bounded sequence in $L^\infty(T)$; hence for some sequence $n_j \to \infty$, f_{n_j} converges in the weak-* topology ($L^\infty(T)$ is the dual of $L^1(T)$) to some function $f(e^{it})$.

Let $re^{i\theta} \in D$, then

$$\begin{aligned} \int f(e^{it})P_r(e^{i(\theta-t)})d\sigma(t) &= \lim_{j\to\infty}\int F(r_{n_j}e^{it})P_r(e^{i(\theta-t)})d\sigma(t) \\ &= \lim_{j\to\infty} F(r_{n_j}re^{it}) = F(re^{i\theta}). \end{aligned}$$

Clearly, the fact that $F = P_r * f$ implies immediately that if $F(z) = \sum a_n r^{|n|}e^{in\theta}$, then $a_n(f) = a_n$.

Note that in the above derivation we used the fact that for any harmonic function u in D and $0 \le r < 1$,

$$\int P_r(e^{i(\theta-t)})u(\rho e^{it})d\sigma(t) = u(\rho re^{i\theta}).$$

This can be verified by considering the representation theorem of harmonic functions in disk: If u is real, continuous on $|z| \le \rho$ and harmonic in $|z| < \rho$, then for $z = \rho_1 e^{i\theta}$, $\rho_1 < \rho$,

$$u(\rho_1 e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} Re[\frac{\rho e^{it}+z}{\rho e^{it}-z}]u(\rho e^{it})dt.$$

Let $\rho_1 = r\rho$ $(0 \le r < 1)$. Then

$$u(r\rho e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} Re[\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}]u(\rho e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} P_r(e^{i(\theta-t)})u(\rho e^{it})dt.$$

For complex harmonic, we consider it as a sum of real part and imaginary part.

Let $C(T)$ be the space of continuous functions on T. Then by F. Riesz' theorem, $(C(T))^* = M(T)$, where $M(T)$ is the space of bounded complex Borel measures on T. Since $C(T)$ is separable, every bounded subset of $M(T)$ is weak* sequentially compact. Note that $L^1(T)$ is contained in $M(T)$, if we identify $f \in L^1$ with the measure $f(e^{ix})d\sigma(x)$.

Lemma 5.1. *Every complex harmonic function F in D has a development $F(re^{i\theta}) = \sum a_n r^{|n|}e^{in\theta}$.*

Proof: Suppose that F is real. Since D is a simply connected region, F has a harmonic conjugate G so that $H = F + iG$ is analytic in D. We write $H(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$\begin{aligned} H(z) = Re(H) &= a_0 + \frac{1}{2}\Big(\sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \overline{a_n} r^n e^{-inx}\Big) \\ &= a_0 + \frac{1}{2}\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \end{aligned}$$

where $a_{-n} = \overline{a_n}$ for $n = 1, 2, \cdots$. If F is complex, then it is linear combination of two real harmonic functions. The desired development of F is the sum of two absolutely convergent series.

Theorem 5.4. *Let F be a function harmonic in the unit disk D. There is a unique measure $\mu \in M(T)$ such that*

$$F(z) = P_r * \mu(e^{i\theta}), \qquad z = re^{i\theta}$$

if and only if

$$A_r = \int |F(re^{i\theta})| d\sigma(\theta) \le K, \quad \forall \quad 0 < r < 1.$$

Moreover, $\|\mu\| = \lim_{r\uparrow 1} A_r$.

Proof: Necessity: If we think of $P_r * \mu(e^{i\theta})$ as a family (with continuous parameter $0 < r < 1$) of functions defined on T, call them f_r, then

$$\|f_r\|_1 = \|P_r * \mu\| \le \|P_r\|_1 \|\mu\|.$$

Therefore, $\{f_r\}$ in bounded in $L^1(T)$. This shows that in order for a harmonic function $F(z)$ in D to be the Poisson integral of some measure it is necessary that

$$\int |F(re^{i\theta})| d\sigma(\theta) \le K, \quad \forall \quad 0 < r < 1.$$

Sufficiency: Given a harmonic function $F(re^{ix})$ in D, by the lemma, one can always write F as a 'power' series:

$$F(re^{ix}) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx},$$

where $a_n r^{|n|}$ is the nth Fourier coefficient of $f_r(e^{ix}) = F(re^{ix})$.

By assumption, $\|f_r\|_1 \le K$, i.e., $\|f_r d\sigma\|_{M(T)} = \|f_r\|_1 \le K \quad \forall r$. Since $C(T)$, as the predual of $M(T)$, is separable normed space (polynomials with rational coefficients are dense in $C(T)$), by Banach-Alaoglu theorem, the closure of $\{f_r d\sigma\}$ in $M(T)$ is weak* sequentially compact. Therefore, there is a subsequence $\{f_{r_j}\}d\sigma$ of $f_r d\sigma$ that converges to some $\mu \in M(T)$ in weak* topology. That is,

$$\int h(e^{-ix}) f_{r_j}(e^{ix}) d\sigma(x) \to \int h(e^{-ix}) d\mu, \qquad j \to \infty$$

for each $h \in C(T)$. In particular, for each n, $a_n(f_{r_j}) \to a_n(\mu)$ as $j \to \infty$. On the other hand, $a_n(f_r) = a_n r^{|n|} \to a_n$ as $r \uparrow 1$. Therefore, $a_n(\mu) = a_n$ for all n.

It follows from the Unicity theorem that μ is uniquely determined by a_n, therefore by F, and that since $a_n(f_r) = a_n r^{|n|} = a_n \mu r^{|n|} = a_n(P_r * \mu)$, $f_r = P_r * \mu$, i.e. $F(re^{ix}) = P_r * \mu(e^{ix})$.

We show that $\|\mu\| = \lim_{r\uparrow 1} A_r$. Note that $\mu = \lim_{j\to\infty} F(r_j e^{ix}) d\sigma(x)$ in the weak* topology of $M(T)$ as the dual of $C(T)$. It follows that $\|\mu\| \le \liminf_{j\to\infty} A_{r_j}$ where $A_{r_j} = \|F(r_j e^{i\cdot})\|_1$. Since A_r increases with r and $A_r \le K$, $\|\mu\| \le \lim_{r\to\infty} A_r$. Furthermore, the inequality cannot be strict. Note that $f_r = P_r * \mu$ and $\|f_r\|_1 \le \|P_r\|_1 \|\mu\|$. Therefore, $A_r = \|f_r\|_1 \le \|\mu\|$ for every $0 < r < 1$. If the inequality were strict, we would have $A_r \le \|\mu\| < \lim_{r\to\infty} A_r$ for $0 < r < 1$, which is impossible.

As to the norm convergence of $\|f_r - \mu\|_{M(T)} \to 0$ as $r \to 0$, if μ is absolutely continuous, then $\mu = f d\sigma$ for some $f \in L^1(T)$. Now $f_r = P_r * \mu$ is indeed $f_r = P_r * f$. Thus, by Fejér's theorem, $\|f_r - f\|_1 \to 0$. That is, $\|f_r - \mu\|_{M(T)} \to 0$.

Theorem 5.5. *Suppose F is harmonic in the unit disk D. If $1 < p \le \infty$, F is the Possion integral of a (unique) function f belonging to $L^p(T)$ if and only if*

$$A_r^p = \int |F(re^{ix})|^p d\sigma(x) \le K < \infty, \quad 0 < r < 1.$$

Moreover, $\|f\|_p = \lim_{r\uparrow 1} A_r$.

Proof: Similar to the proof of the previous theorem.

6. Convergence a.e.

Theorem 6.1. *If $f \in L^1(T)$, then the symmetric derivative of indefinite integral of f*

$$\lim_{t\to 0} \frac{1}{2t} \int_{x-t}^{x+t} f(u)du = f(x), \quad a.e.$$

Furthermore, for each x where this holds,

$$\lim_{r\uparrow 1} (p_r * f)(x) = f(x).$$

Proof: The first part is simply the Lebesgue differentiation Theorem: if $f \in L^1$, then for a.e. x,

$$\lim_{Q\downarrow x} \frac{1}{|Q|} \int_Q f(u)du = f(x),$$

where Q's are intervals centered at x.

Let x be a point where the above limit holds, i.e., at x

$$\int_0^t (f(x+u) + f(x-u) - 2f(x))du = o(t).$$

Let $G(t) = \int_0^t (f(x+u) + f(x-u) - 2f(x))du$. Given any $\epsilon > 0$, there is $\delta > 0$ so that if $0 \leq t \leq \delta$, then $|G(t)| \leq \epsilon t$. Also note that $G(t)$ is absolutely continuous so that $G'(t) = f(x+t) + f(x-t) - 2f(x)$ for a.e. t. We write

$$\begin{aligned} (p_r * f)(x) &= \frac{1}{2\pi}\int_0^\pi (f(x+u) + f(x-u) - 2f(x))p_r(u)du \\ &= \frac{1}{2\pi}\Big(\int_0^\delta + \int_\delta^\pi\Big)(f(x+u) + f(x-u) - 2f(x))p_r(u)du = I_1 + I_2. \end{aligned}$$

We consider I_1 first. We have

$$\begin{aligned} 2\pi|I_1| &= |\int_0^\delta G'(u)p_r(u)du| \\ &= |G(\delta)p_r(\delta) - \int_0^\delta G(u)p_r'(u)du| \\ &\leq |G(\delta)p_r(\delta)| + \int_0^\delta |G(u)||p_r'(u)|du \\ &\leq \epsilon\Big(|\delta p_r(\delta)| + \int_0^\delta u|p_r'(u)|du\Big). \end{aligned}$$

Note that $\delta p_r(\delta) > 0$. Using integration by parts, we have

$$|\delta p_r(\delta)| = \delta p_r(\delta) = \int_0^\delta up_r'(u)du + \int_0^\delta p_r(u)du.$$

Furthermore, since $p_r'(u)$ keeps constant sign (negative) as $x > 0$,

$$2\pi|I_1| = \epsilon(\int_0^\delta up_r'(u)du + \int_0^\delta p_r(u)du + \int_0^\delta u|p_r'(u)|du) = \epsilon\int_0^\delta p_r(u)du.$$

The last integral goes to 0 as $r \uparrow 1$.

The following lemma can be used to give an alternative proof of the above theorem:

Lemma 6.1. *Let $e_n(u)$ be an even approximate identity. Then for any bounded function f, at the point x where $f(x+)$ and $f(x-)$ exist, we have*

$$\lim_{n\to\infty}\int_{-\pi}^\pi f(x-u)e_n(u)d\sigma(u) \to \frac{f(x+) + f(x-)}{2}.$$

In particular, if $\lim_{u\to x} f(u) = L$, *then*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x-u)e_n(u)d\sigma(u) \to L.$$

Proof: If e_n is even, then

$$\begin{aligned} I &= \int_{-\pi}^{\pi} f(x-u)e_n(u)d\sigma(u) - \frac{f(x+)+f(x-)}{2} \\ &= \int_{0}^{\pi}\Big(f(x+u)+f(x-u)-f(x+)-f(x-)\Big)e_n(u)d\sigma(u). \end{aligned}$$

Given $\epsilon > 0$, there exists $\delta > 0$ such that for $0 < u \leq \delta$, $|f(x+u) - f(x+)| < \epsilon$ and $|f(x-u) - f(x-)| < \epsilon$. We write

$$I = \left(\int_0^{\delta} + \int_{\delta}^{\pi}\right)(f(x+u) - f(x+) + f(x-u) - f(x-))e_n(u)d\sigma(u) = I_1 + I_2.$$

For I_1, we have

$$|I_1| \leq 2\epsilon\int_0^{\pi} e_n(u)d\sigma(u) = 2\epsilon.$$

For I_2, we have

$$|I_2| \leq 4M\int_{\delta}^{\pi} e_n(u)d\sigma(u) < \epsilon,$$

for sufficiently large n.

An alternative proof of the above theorem.

Proof: First, we make the following assumptions successively:

(1) We may assume that $x = 0$ is the point where

$$\lim_{t\to 0}\frac{1}{2t}\int_{x-t}^{x+t} f(u)du = f(x).$$

That is,

$$\lim_{t\to 0}\frac{1}{2t}\int_{-t}^{t} f(u)du = f(0).$$

Assume that the limit holds for f at $x = a$. Then $g(x) = f(x+a)$ satisfies

$$\lim_{t\to 0}\frac{1}{2t}\int_{-t}^{t} g(u)du = g(0).$$

If the theorem is proved for g at 0, then $(p_r * g)(0) \to g(0)$ is simply $(p_r * f)(a) \to f(a)$.

(2) We may also assume that $f(0) = 0$. Let $g(x) = f(x) - f(0)$. Then $g(0) = 0$. If the theorem is proved for g, then $(p_r * g)(0) \to 0)$ is simply $(P_r * (f(\cdot) - f(0))(0) = (p_r * f)(0) - f(0) \to 0$, which is $(p_r * f)(0) \to f(0)$.

(3) Finally, we may assume that $\int f(x)d\sigma(x) = 0$. Let g be a smooth function with $\int g = \int f$, and vanishing on a neighborhood of $x = 0$ (maintaining the above two assumptions). If the theorem is proved for $f - g$, then it follows for f.

Now we proceed to prove the theorem: If $f \in L^1(T)$ with $\int f = 0$, and

$$\lim_{t\to 0} \frac{1}{2t}\int_{-t}^{t} f(u)du = 0,$$

then

$$\lim_{r\uparrow 1}(p_r * f)(0) = 0.$$

Proof: We first prove that

$$q_r(x) = r^{-1}(-\sin x p_r'(x))$$

is an approximate identity. Furthermore, if we define $F(t) = \int_{-\pi}^{t} f(x)dx$, then we can write $p_r * f(0)$ as, using integration by parts and $\int f = 0$,

$$p_r * f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p_r(x)f(x)dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} p_r'(x)F(x)dx.$$

Since p_r' is odd, the last integral can be written as

$$\begin{aligned} -\frac{1}{2\pi}\int_{-\pi}^{\pi} p_r'(x)F(x)dx &= -\frac{1}{2\pi}\int_{0}^{\pi} p_r'(x)(F(x) - F(-x))dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} rq_r(x)\frac{x}{\sin x}\frac{F(x) - F(-x)}{2x}dx. \end{aligned}$$

Note that the function $\frac{x}{\sin x}\frac{F(x) - F(-x)}{2x}$ is bounded (since $f \in L^1$, F is bounded. In addition, $\frac{F(x) - F(-x)}{2x} \to 0$ as $x \to 0$). By the lemma, we have that

$$\begin{aligned} p_r * f(0) &= \frac{1}{\pi}\int_{0}^{\pi} rq_r(x)\frac{x}{\sin x}\frac{F(x) - F(-x)}{2x}dx \\ &\to \lim_{x\to 0}(\frac{x}{\sin x}\frac{F(x) - F(-x)}{2x}) = 0, \quad r \uparrow 1. \end{aligned}$$

Lemma 6.2. *The Fejér kernel*

$$K_n(x) = \frac{1}{n}|\sum_{0}^{n-1} e^{ikx}|^2 = \frac{1}{n}(\frac{sin\frac{1}{2}nx}{sin\frac{1}{2}x})^2$$

has a bell-shaped majorant

$$K_n^*(x) = \frac{2\pi^2 n}{1 + n^2 x^2}.$$

Proof: The first formula for K_n gives $K_n(x) \leq n$ (used for smaller x). By Jordan's inequality, the second formula leads to $K_n(x) \leq \frac{\pi^2}{nx^2}$ (used for large x). Combining these two gives $K_n(x) \leq K_n^*(x)$, where $K_n^*(x) = \frac{2\pi^2 n}{1+n^2x^2}$ (consider $|x| \leq \frac{1}{n}$ and $|x| > \frac{1}{n}$ separately).

Theorem 6.2. *Let $f \in L^1$ and let x be such a point where*

$$\lim_{t\to 0} \frac{1}{t} \int_0^t |f(x+u) + f(x-u) - 2L|du = 0$$

for some L (Note that if $f \in L^1$, then $L = f(x)$ for a.e. x). Then $K_n * f(x) \to L$ as $n \to \infty$.

Proof: First, we write

$$\begin{aligned}
|K_n * f(x) - L| &= \Big|\frac{1}{2\pi}\int_0^\pi K_n(t)(f(x+t) + f(x-t) - 2L)dt\Big| \\
&= \Big|\frac{1}{2\pi}\Big(\int_0^{\frac{2\pi}{n}} + \int_{\frac{2\pi}{n}}^\pi\Big)K_n(t)(f(x+t) + f(x-t) - 2L)dt\Big| \\
&\leq \frac{1}{2\pi}\Big(\int_0^{\frac{2\pi}{n}} + \int_{\frac{2\pi}{n}}^\pi\Big)K_n(t)|f(x+t) + f(x-t) - 2L|dt = I_1 + I_2.
\end{aligned}$$

It is interesting to note that the splitting point varies with n.

Let $H(t) = \int_0^t |f(x+t) + f(x-t) - 2L|dt$. Then $\frac{H(t)}{t} \to 0$ as $t \to 0$. For I_1 we have

$$2\pi I_1 \leq nH(\frac{2\pi}{n}) \to 0, \qquad n \to \infty.$$

For I_2 we have

$$\begin{aligned}
2\pi I_2 &\leq \int_{\frac{2\pi}{n}}^\pi K_n^*(t)|f(x+t) + f(x-t) - 2L|dt \\
&= K_n^*(\pi)H(\pi) - K_n^*(\frac{2\pi}{n})H(\frac{2\pi}{n}) - \int_{\frac{2\pi}{n}}^\pi H(t)(K_n^*)'(t)dt.
\end{aligned}$$

Since $H(\pi)$ is finite, $K_n^*(\pi)H(\pi) \to 0$ as $n \to 0$. We also have that

$$K_n^*(\frac{2\pi}{n})H(\frac{2\pi}{n}) \leq \frac{n}{4}H(\frac{2\pi}{n}) \to 0, \quad n \to \infty.$$

As for the last term, we have

$$\begin{aligned} & - \int_{\frac{2\pi}{n}}^{\pi} H(t)(K_n^*)'(t)dt \\ = & \int_{\frac{2\pi}{n}}^{\pi} H(t)\frac{4\pi^2 n^3 t}{(1+n^2t^2)^2}dt \\ \leq & \int_{\frac{2\pi}{n}}^{\pi} H(t)\frac{4\pi^2}{nt^3}dt. \end{aligned}$$

We show that the last integral tends to 0. Given $\epsilon > 0$, there exists $\delta > 0$ such that $H(t) < \epsilon t$ if $0 < t < \delta$. Then

$$\frac{1}{n}\int_{\frac{2\pi}{n}}^{\pi} \frac{H(t)}{t^3}dt \leq \frac{1}{n}\int_{\frac{2\pi}{n}}^{\delta} \frac{\epsilon}{t^2}dt + \frac{1}{n}\int_{\delta}^{\pi} \frac{H(t)}{t^3}dt.$$

The first term on the right is majorized by ϵ (simply integrate), while the last term clearly tends to 0 as $n \to \infty$.

Theorem 6.3. *The boundedness of f in the lemma 6.1 is indispensable.*

Proof: We define e_n as an (even) approximate identity with a "tail":

$$2\pi e_n(x) = \begin{cases} n - \sqrt{n} & 0 \leq x < \frac{1}{n}, \\ 0 & \frac{1}{n} \leq x < (\pi - \frac{1}{n}), \\ \sqrt{n} & (\pi - \frac{1}{n}) \leq x < \pi. \end{cases}$$

Clearly, $e_n(x) \geq 0$, $\displaystyle\int e_n(x)dx = 1$, and for any $\delta > 0$, $\int_{|x|>\delta} e_n(x)dx = \frac{1}{\sqrt{n}} \to 0$.

Let $f(x)$ be an even function defined as

$$f(x) = \begin{cases} 0 & 0 \leq x < \pi - 1 \\ \sqrt{n} & \pi - \frac{1}{n} \leq x < \pi - \frac{1}{n+1}, n = 1, 2, \cdots. \end{cases}$$

Then $\displaystyle\int f = \sum \frac{\sqrt{n}}{n(n+1)} < \infty$ so that $f \in L^1$. But $e_n * f(0) = \int e_n(x)f(x)dx \geq 1$ for all n. We do not have $e_n * f(0) \to f(0) = 0$. In this example, the coincidence of the tail of f and that of e_n makes $\displaystyle\int e_n(x)f(x)dx$ big.

Theorem 6.4. *A bounded analytic function F in D has radial limits at a.e. boundary point.*

Proof: We write $F = G + iH$. Since G is harmonic and bounded, there is a bounded function g on T such that $G(re^{ix}) = (p_r * g)(e^{ix})$. Note that $(p_r * g)(e^{ix}) \to g(e^{ix}), a.e.$ That is, $G(re^{ix}) \to g(e^{ix}), a.e.$, as $r \uparrow 1$. Similarly, there is a bounded function h on T so that $H(re^{ix}) \to h(e^{ix})$. Thus, $F(re^{ix}) \to g(e^{ix}) + ih(e^{ix})$ as $r \uparrow 1$.

Theorem 6.5. *If μ is a singular measure on T, then $P_r * \mu$ tends to* 0 *a.e. as r increases to* 1.

Proof: Let $x = 0$ be a point where

$$\lim_{u\to 0}\frac{\mu([-u,u))}{2u} = 0.$$

We show that

$$(p_r * \mu)(0) = \int_{-\pi}^{\pi} p_r(u)d\mu(u) \to 0, \quad r \uparrow 1.$$

Let $F(t) = \displaystyle\int_{-\pi}^{t} d\mu(v)$. Then, by Fubini's theorem,

$$\begin{aligned}
\int_{-\pi}^{\pi} p_r'(u)F(u)du &= \int_{-\pi}^{\pi} p_r'(u)\int_{-\pi}^{u} d\mu(v)du \\
&= \int_{-\pi}^{\pi}\int_{v}^{\pi} p_r'(u)dud\mu(v) \\
&= \int_{-\pi}^{\pi}(p_r(\pi) - p_r(v))d\mu(v) \\
&= p_r(\pi)\mu([-\pi,\pi)) + \int_{-\pi}^{\pi} p_r(v)d\mu(v).
\end{aligned}$$

It follows that

$$\int_{-\pi}^{\pi} p_r(v)d\mu(v) = \int_{-\pi}^{\pi} p_r'(u)F(u)du - p_r(\pi)\mu([-\pi,\pi)).$$

For the last integral, noting that p_r' is odd, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} p_r'(u)F(u)du &= \int_{0}^{\pi} p_r'(u)(F(u) - F(-u))du \\
&= \int_{0}^{\pi} 2up_r'(u)\frac{\mu([-u,u))}{2u}du \\
&\to \lim_{u\to 0}\frac{\mu([-u,u))}{2u} = 0, \quad r \uparrow 1.
\end{aligned}$$

7. Herglotz' Theorem

Definition 7.1. *A complex sequence $\{u_n\}_{n=-\infty}^{\infty}$ is called positive definite if*

$$\sum_{m,n} u_{m-n}c_m\overline{c_n} \geq 0$$

for every sequence $\{c_n\}_{n=-\infty}^{\infty}$ such that $c_n = 0$ except for a finite number of n.

Theorem 7.1. *Let μ be any positive measure on $[0, 2\pi)$. Set*

$$u_n = \int e^{-inx}d\mu(x).$$

Then u_n is positive definite.

Proof: Note that

$$\begin{aligned}\int |\sum c_m e^{-imx}|^2 d\mu(x) &= \int (\sum c_m e^{-imx})(\sum \overline{c_n} e^{inx}) d\mu(x)\\ &= \int \sum_{m,n} c_m \overline{c_n} e^{-i(m-n)x} d\mu(x)\\ &= \sum_{m,n} c_m \overline{c_n} u_{m-n}.\end{aligned}$$

The result to be proved is that these are the only positive definite sequences.

Theorem 7.2 (Herglotz)**.** *Every positive definite sequence u_n is the sequence of the Fourier-Stieltjes coefficients of a positive measure.*

Proof: Use the given sequence u_n to define a linear functional F on trigonometric polynomials. For each trigonometric polynomial $\varphi = \sum c_n e^{inx}$, define

$$F(\varphi) = \sum c_n u_n.$$

Clearly, F is a linear (over complex scalar) functional defined on a dense subspace of $C(T)$.

Prove that F is nonnegative on every nonnegative trigonometric polynomial. Note that

$$|\varphi|^2 = (\sum c_m e^{imx})(\sum \overline{c_n} e^{-inx}) = \sum_{m,n} c_m \overline{c_n} e^{i(m-n)x}$$

and that u_n is positive definite. Thus

$$F(|\varphi|^2) = F(\varphi\overline{\varphi}) = \sum_{m,n} c_m \overline{c_n} u_{m-n} \geq 0.$$

This means that $F(\psi) \geq 0$ for any trigonometric polynomial ψ that can be written as $\psi = |\varphi|^2$ for some trigonometric polynomial φ. Note that every nonnegative trigonometric polynomial has this form. Thus F is a linear functional on trigonometric polynomials that is nonnegative on every nonnegative trigonometric polynomial.

Prove that F is a (complex-valued) continuous linear functional on the space of real trigonometric polynomials. It follows from the above argument that for any two trigonometric polynomials φ and ψ with $\varphi \leq \psi$, $F(\varphi) \leq F(\psi)$. In particular, if we denote $k = F(1)$, then (choose $\psi = 0$ and 1, respectively) for any $0 \leq \varphi \leq 1$, $0 \leq F(\varphi) \leq k$. Let φ be a trigonometric polynomial with $|\varphi| \leq 1$. Then $0 \leq 1 + \varphi \leq 2$. Thus $0 \leq F(1) + F(\varphi) = F(1 + \varphi) \leq 2k$ and so $|F(\varphi)| \leq k$. Thus F is a (complex-valued) continuous linear functional on the space of real trigonometric polynomials.

Extend F to a continuous linear functional on $C(T)$ - the space of the complex valued continuous functions. Since the space of real trigonometric polynomials is dense in the space of all real continuous function and F is continuous on the space of real trigonometric polynomials, F can be extended (by limit) to a continuous linear functional on all real continuous functions. Further, for complex valued continuous function f, define $F(f) = F(Ref) + iF(Imf)$. Then F is a continuous linear functional on $C(T)$. Also note that this definition, when restricted on trigonometric polynomials, coincides with the original definition for F.

Use F to find the desired measure μ. Since $F \in C(T)^*$, by the Riesz representation theorem there exists a (complex, in general) measure μ such that

$$F(f) = \int f d\mu(x)$$

for all $f \in C(T)$. Moreover, looking at how μ is constructed, we see that

$$\mu(E) = \int_E \overline{g} d\iota$$

for some $g \in L_\infty(T)$ with $\|g\|_\infty = 1$ and ι a positive Borel measure on T.

Prove that g is positive and so μ is indeed a positive measure. Since each nonnegative continuous function is the uniform limit of nonnegative trigonometric polynomials (for example, the Fejér means of its Fourier series), F is nonnegative on all nonnegative continuous functions. That is,

$$F(f) = \int_X f\overline{g} d\iota \geq 0$$

for all nonnegative $f \in C(T)$. It follows that $g \geq 0$ a.e. and so μ is a positive measure.

Prove that $\hat{\mu}(n) = u_n$. It follows from the previous proofs (let $\varphi = e^{inx}$) that $\hat{\mu}(n) = \int e^{-inx} d\mu(x) = F(e^{inx}) = u_n$.

An alternative proof of Herglotz' theorem.

Define A as the space of functions $f \in C(T)$ with absolutely convergent Fourier series. That is, if $f \sim \sum a_n(f)e^{inx}$, then $\sum |a_n(f)|$ converges (so that the Fourier series of f converges uniformly to f).

Claim: A is a Banach algebra under multiplication (the product of f and g is defined as $f\overline{g}$) in the norm inherited from l^1.

Proof: Let $f \in A$. $\|f\| = \|f\|_A = \|\{a_n(f)\}\|_{l^1}$. Then A is a normed linear space. We prove that A is complete. Let f_k be a Cauchy sequence in A, that is, $\{a_n(f_k)\}$ is a Cauchy sequence in l^1. Assume that this sequence converges to $\{b_n\} \in l^1$. Let $f = \sum b_n e^{inx}$. Then $f \in A$ and $\|f_k - f\| = \|\{a_n(f_k) - b_n\}\|_{l^1} \to 0$.

To prove that A is a Banach algebra, we prove first that if both f and g are in A, then so is $f\overline{g}$. Note that if $f(e^{ix}) = \sum a_k(f)e^{ikx}$ and $g(e^{ix}) = \sum a_j(g)e^{ijx}$, then

$$\begin{aligned} f(e^{ix})\overline{g(e^{ix})} &= (\sum a_k(f)e^{ikx})(\sum \overline{a_j(g)}e^{-ijx}) \\ &= \sum_{k,j} a_k(f)\overline{a_j(g)}e^{i(k-j)x} = \sum c_l e^{ilx}, \end{aligned}$$

where

$$c_l = \sum_k a_k(f)\overline{a_{k-l}(g)}.$$

If $a_k(f), a_j(g) \in l^1$, then c_l converges absolutely for every l. Moreover, since $\{c_l\}$ is the convolution of $\{a_k(f)\}$ and $\{a_j(g)\} \in l^1$, $\{c_l\} \in l^1$ (like that in L^1) and $fg \in A$. Secondly, we verify that $\|f\overline{g}\| \le \|f\|\|g\|$. Note that the inequality actually says that $\|a * b\|_{l^1} \le \|a\|_{l^1}\|b\|_{l^1}$ for $a, b \in l^1$. But it is true just like in L^1.

Claim: Define for all $\varphi = \sum c_n e^{inx} \in A$,

$$F(\varphi) = \sum c_n u_n.$$

Then $F(|\varphi|^2) \ge 0$ for each $\varphi \in A$.

We need to justify this definition first. Note that $|u_n| \le u_0$ for all n. Thus $\sum c_n u_n$ converges absolutely so that $F(\varphi)$ is well-defined for all $\varphi \in A$.

Next, we show $F(|\varphi|^2) \ge 0$ for all $\varphi \in A$. Note that $|\varphi|^2 = \varphi\overline{\varphi} \in A$. By definition of F,

$$F(|\varphi|^2) = \sum_{k,j} a_k(\varphi)\overline{a_j(\varphi)}u_{k-j}.$$

(The coefficient of u_0 is $\sum |a_k(\varphi)|^2$). If $a_k(\varphi)$'s are zeros except for finitely many k, then $\sum_{k,j} a_k(\varphi)\overline{a_j(\varphi)}u_{k-j} \ge 0$. Hence $F(|\varphi|^2) \ge 0$ as long as the sum that evaluates $F(|\varphi|^2)$ converges, which is indeed the case because u_n's are bounded.

Claim: If $\psi \in A$ and $\psi > 0$ (strictly positive!), then $\psi = |\varphi|^2$ for some $\varphi \in A$. Hence, $F(\psi) \ge 0$ for all $\psi \in A$ and $\psi > 0$.

Proof: Note that $\sqrt{z}$ is analytic on the right-half (open) plane that contains the range of ψ ($\psi > 0$) and ψ has absolutely convergent Fourier series. By the Weiner-Levy theorem, $\sqrt{\psi}$ has absolutely convergent Fourier series. That is, $\sqrt{\psi} \in A$.

Claim: $|F(\varphi)| \leq M\|\varphi\|_A$ for $\varphi \in A$ and $|\varphi| \leq 1$.

Proof: We have shown that $F(\psi) \geq 0$ for all $\psi \in A$ and $\psi > 0$. It follows that for any two $\varphi, \varphi_1 \in A$ with $\varphi < \varphi_1$, $F(\varphi) \leq F(\varphi_1)$. In particular, if we denote $k = F(1)$, then (choose $\varphi_1 = 0$ and 1, respectively) for any $0 < \varphi < 1$, $0 \leq F(\varphi) \leq k$. If $-1 < \varphi < 1$ with $\varphi = 0$ somewhere (i.e. $|\varphi| < 1$), then $0 < 1 + \varphi < 2$, and $0 \leq F(1 + \varphi) \leq 2k$. That is, $|F(\varphi)| \leq k$. For $\varphi \in A, |\varphi| \leq 1$, we consider $\varphi/2$. Then $|\varphi/2| < 1$ and $|F(\varphi/2)| \leq k$, i.e., $|F(\varphi)| \leq 2k$ for $\varphi \in A$ and $|\varphi| \leq 1$. Therefore, F is a (complex-valued) continuous linear functional on A. The rest of the proof is exactly the same as that in the first proof.

8. Hausdorff-Young Inequality

Theorem 8.1 (Riesz-Thorin Convexity Theorem)**.** *Assume that $p_0 \neq p_1$, $q_0 \neq q_1$ and let T be a linear operator such that*

$$T : L_{p_0}(U, d\mu) \to L_{q_0}(V, d\nu)$$

with norm M_0, and that

$$T : L_{p_1}(U, d\mu) \to L_{q_1}(V, d\nu)$$

with norm M_1. Then

$$T : L_p(U, d\mu) \to L_q(V, d\nu)$$

with norm $M_\theta \leq M_0^{1-\theta} M_1^\theta$, provided that $0 \leq \theta \leq 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}; \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Proof: (See [4])

Theorem 8.2 (Hausdorff-Young)**.** *For any $f \in L_p(T)$, $1 \leq p \leq 2$,*

$$\|\hat{f}\|_q \leq \|f\|_p,$$

where q is the exponent conjugate to p and, of course, $\|\hat{f}\|_q$ is the norm of the sequence of Fourier coefficients of f in l_q.

Proof: The Hausdorff-Young inequality is a simple consequence of the Riesz-Thorin convexity theorem.

Let $Tf = \hat{f}$. Note that Tf is the sequence of Fourier coefficients of f. It is trivial that

$$\|Tf\|_\infty \leq \|f\|_1.$$

In addition, Bessel's inequality gives

$$||Tf||_2 \leq ||f||_2.$$

Given p with $1 \leq p \leq 2$, let $0 \leq \theta \leq 1$ be such that

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}.$$

By the Riesz-Thorin theorem, we have

$$||Tf||_q \leq ||f||_p, \ \forall f \in L_p,$$

where q is given by

$$\frac{1}{q} = 1 - \frac{1}{p}.$$

It is worth noting that we couldn't get the best constant in the Hausdorff- Young inequality. Beckner proved (Annals of Math, 102(1975)) for the Fourier transforms on R that

$$||\hat{f}||_q \leq \sqrt{\frac{p^{1/p}}{q^{1/q}}}||f||_p.$$

The following proof of the Hausdorff-Young inequality is due to A.P.Calderon and A. Zygmund. It suffices to show that for any trigonometric polynomial f with Fourier coefficients $c = (c_n)$ and $||f||_p = 1$ we have $||c||_q \leq 1$. Using the duality, we see that it suffices to show that

$$|\sum c_n d_n| \leq 1$$

for every sequence d with $||d||_p = 1$.

Put $f(t) = F(t)^{1/p}E(t)$ for $t \in T$ such that $F(t) = |f(t)|^p \geq 0$ and $|E(t)| = 1$. ($E(t) = exp\{iarg(f(t))\}$. In case $f(t) = 0$, simply define $E(t) = 1$). Similarly, put $d_n = D_n^{1/p}e_n$ with $D_n \geq 0$ and $|e_n| = 1$.

Using these functions we write $\sum c_n d_n$ as

$$\sum c_n d_n = \sum D_n^{1/p} e_n \int F(t)^{1/p}E(t)e^{-int}dt.$$

Introducing the complex variable z, we define the function

$$Q(z) = \sum D_n^z e_n \int F(t)^z E(t)e^{-int}dt.$$

Using the Lebesgue Dominated Convergence Theorem we can prove that $Q(z)$ is analytic in $Rez > 0$.

Since the sum has only finitely many terms, each one (as function of z) is bounded in the strip $\frac{1}{2} \leq Rez \leq 1$. Hence $Q(z)$ is bounded in this strip with bound depending on $d_n's$ and f.

For $Rez = 1$, we have

$$|Q(1+it)| \leq \sum D_n \int F(t)dt = 1.$$

For $Rez = \frac{1}{2}$, the Schwarz inequality gives

$$|Q(\frac{1}{2}+i\theta)| \leq (\sum D_n)^{1/2} (\sum |\int F(t)^{\frac{1}{2}+i\theta} E(t) e^{-int} dt|^2)^{1/2}.$$

The integral is the Fourier coefficient of $F(t)^{\frac{1}{2}+i\theta} E$. Bessel's inequality gives that

$$(\sum |\int F(t)^{\frac{1}{2}+i\theta} E(t) e^{-int} dt|^2)^{1/2} \leq \|f^{p/2}\|_2 = (\|f\|_p)^{p/2} = 1.$$

Therefore

$$|Q(\frac{1}{2}+i\theta)| \leq 1.$$

$Q(z)$ is analytic and bounded in the strip $\frac{1}{2} \leq Rez \leq 1$ (with bound depending on $d_n's$ and f) and bounded by 1 on the lines $Rez = \frac{1}{2}$ and $Rez = 1$. By Hadamard's three-lines theorem, $|Q(z)| \leq 1$ for all z throughout the strip. In particular, taking $z = \frac{1}{p}$ in $Q(z)$, we have $|\sum c_n d_n| \leq 1$ for every sequence d with $\|d\|_p = 1$.

Theorem 8.3. *Let f be a summable function whose coefficient sequence is in l^p, $1 < p < 2$. Show that $f \in L^q$ and $\|f\|_q \leq \|\hat{f}\|_p$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof: Let $1 < p < 2$ and let $c = (c_k)_{k=-\infty}^{\infty} \in l^p$. For $t \in (-\pi, \pi]$, we define

$$(T^* c)(t) = \sum c_k e^{ikt}.$$

We must show that T^* is well-defined in the sense that

$$s_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}$$

converges to a function $f(t)$ on $(-\pi, \pi]$ in the norm of L^q, where q is the exponent conjugate of p.

For any $h \in L^p$, by the Hölder and Hausdorff-Young inequalities,

$$\begin{aligned} |\int_{-\pi}^{\pi} h(t)\overline{s_n(t)}d\sigma| &= |\sum_{k=-n}^{n} \hat{h}(k)\overline{c_k}| \\ &\leq (\sum_{k=-n}^{n} |\hat{h}(k)|^q)^{1/q}(\sum_{k=-n}^{n} |c_k|^p)^{1/p} \\ &\leq ||\overline{h}||_q||c||_p \leq ||h||_p||c||_p. \end{aligned}$$

This implies that $||s_n||_q \leq ||c||_p$ for all n. Note that this is valid for any $c \in l^p$. We have

$$||s_m - s_n||_q = ||\sum_{n<|k|\leq m} c_k e^{ikx}||_q \leq \sum_{n<|k|\leq m} |c_k|^p.$$

Therefore, s_n is a Cauchy sequence in L^q and hence there exists an $f \in L^q$ so that $||s_n - f||_q \to 0$. We simply define $(T^*c)(t) = f(t)$. Note that T^* is an adjoint operator to T, the finite Fourier transform, in the sense that

$$< T(h), c >=< h, T^*(c) >,$$

for all $h \in L^p$ and $c \in l^p$, where $< T(h), c >= \sum \hat{h}(k)\overline{c_k}$ and $< h, T^*(c) >= \displaystyle\int h(t)\overline{f}(t)dt$ with f defined as the L^q limit of s_n.

Moreover, for each k and for any $n > |k|$, by Hölder's inequality,

$$|\hat{f}(k) - c_k| = |\int (f(t) - s_n(t))e^{-ikt}d\sigma| \leq ||f - s_n||_q.$$

Therefore, $\hat{f}(k) = c_k$.

Remarks:

(1) The case $p = 2$ is the theorem of Riesz-Fischer.
(2) The case $p = 1$, to every $c \in l^1$ we may assign the continuous function $f(t) = \sum c_k e^{ikt}$. Since the series converges uniformly, $c_k = \hat{f}(k)$ and $||f||_C \leq ||c||_1$.
(3) The restriction of the theorem to $1 \leq p \leq 2$ is essential. For there is a sequence $c \in l^q$ for all $q > 2$ and yet is not the finite Fourier transform of any function in L^1.

The series

$$\sum \pm n^{-1/2} \cos nx,$$

with a suitable choice of signs, is a desired example as shown by the following theorem: If $\sum(a_n^2 + b_n^2)$ diverges, then almost all the series

$$\sum r_n(t)(a_n \cos nx + b_n \sin nx)$$

are not Fourier series (because almost all the series are almost everywhere non-Fejér summable).

Theorem 8.4. *The restriction of the Hausdorff-Young inequality to* $1 \leq p \leq 2$ *is essential, for there is a continuous function* $f \in C$ *(hence* $f \in L^p$ *for all* $p > 0$*) such that* $\|\hat{f}\|_q = \infty$ *for all* $q < 2$*. Therefore, it is impossible that for some* $p > 2$*, we would have* $\|\hat{f}\|_q \leq \|f\|_p$ *for* $f \in L_p$*.*

Proof: The construction of the desired function follows from the following theorem (see [5]): If $\sum a_n^2 + b_n^2 < \infty$, then almost all series

$$\sum r_n(t)(a_n \cos nx + b_n \sin nx)$$

converges almost everywhere in $x \in [0, 2\pi]$. If $\sum (a_n^2 + b_n^2)(logn)^{1+\epsilon} < \infty$ for some $\epsilon > 0$, then almost all series

$$\sum r_n(t)(a_n \cos nx + b_n \sin nx)$$

converges uniformly and so are Fourier series of continuous functions.

The series

$$\sum \pm \frac{\cos nx}{n^{1/2} \log^2 n}$$

is, for a suitable choice of signs, a case in point.

Theorem 8.5 (Hadamard's three-lines theorem)**.** *Assume that* $f(z)$ *is analytic on the open strip* $0 < Rez < 1$ *and bounded (by B) and continuous on the closed strip* $0 \leq Rez \leq 1$*. If* $|f(it)| \leq M_0$ *and* $|f(1 + it)| \leq M_1$*,* $-\infty < t < \infty$*, then we have* $|f(\theta + it)| \leq M_0^{1-\theta} M_1^{\theta}$*.*

Proof: We assume first that $M_0 = M_1 = 1$. We have to prove that $|f(z)| \leq 1$ for all z in the strip.

For each $\epsilon > 0$, we define

$$h_\epsilon(z) = \frac{1}{1 + \epsilon z}, \quad z \in \text{the strip.}$$

Since $Re(1 + \epsilon z) \geq 1$ in the closed strip, we have $|h_\epsilon| < 1$ in the closed strip, so that

$$|f(z)h_\epsilon(z)| \leq 1$$

for z in the boundaries of the strip. Also $|1 + \epsilon z| \geq \epsilon|y|$, so that

$$|f(z)h_\epsilon(z)| \leq \frac{B}{\epsilon|y|}, \quad z \in \text{the closed strip.}$$

Let R be the rectangle cut off from the closed strip by the lines $y = \pm B/\epsilon$. Since $|fh_\epsilon| \leq 1$ on the boundaries of R, $|fh_\epsilon| \leq 1$ on R, by the maximum modulus principle. But the above also shows that $|fh_\epsilon| \leq 1$ on the rest of the closed strip. Thus $|fh_\epsilon| \leq 1$ throughout the closed strip. If we fix z in the strip and let $\epsilon \to 0$, we obtain $|f(z)| \leq 1$.

We now turn to the general case. Put

$$g(z) = M_0^{1-z} M_1^z,$$

where $M_i^{\zeta} = exp\{\zeta \log M_i\}$ for complex ζ. Then $g(z)$ is entire, g has no zero, $1/g$ is bounded in the closed strip,

$$|g(it)| = M_0, \qquad |g(1+it)| = M_1,$$

and hence f/g satisfies our previous assumptions. Thus $|f/g| \leq 1$ in the strip, and this gives $|f(\theta + it)| \leq M_0^{1-\theta} M_1^{\theta}$ for all $0 \leq \theta \leq 1$.

An Alternative Proof:

Let $\epsilon > 0$ and $\lambda \in R$. Define

$$F_\epsilon(z) = exp\{\epsilon z^2 + \lambda z\} F(z).$$

Then

$$F_\epsilon(z) \to 0, \text{ as } Imz \to \pm\infty$$

and

$$|F_\epsilon(it)| \leq M_0, \ |F_\epsilon(1+it)| \leq M_1 e^{\epsilon+\lambda}.$$

By the Phragmen-Lindelöf principle we therefore obtain

$$|F_\epsilon(z)| \leq max\{M_0, M_1 e^{\epsilon+\lambda}\}.$$

That is,

$$|F(\theta + it)| \leq exp\{-(\theta^2 - t^2)\} max\{M_0 e^{-\theta\lambda}, M_1 e^{(1-\theta)\lambda+\epsilon}\}.$$

This holds for any fixed θ and t. Letting $\epsilon \to 0$ we conclude that, if $\rho = exp\{\lambda\}$,

$$|F(\theta + it)| \leq max\{M_0 \rho^{-\theta}, M_1 \rho^{1-\theta}\}.$$

9. A Theorem of Minkowski

Let

$$T^2 = \{(e^{i2\pi x}, e^{i2\pi y}) : x, y \in R\}.$$

T^2 is called the 2-dimensional torus, which is the Cartesian product of the unit circle $T = \{e^{i2\pi x} : x \in R\}$.

Let (m, n) be a lattice (integer coordinates) point in the plane and let $f(x, y)$ be a summable function on the unit square

$$E = \{(x, y) : 0 < x < 1, 0 < y < 1\}$$

with extension to R^2 periodically. Define the Fourier coefficients

$$a_{m,n}(f) = \int\int_E f(x, y) e^{2\pi i(mx+ny)} dxdy.$$

We may prove the Parseval relation on $L^2(T^2)$. For any trigonometric polynomial

$$p(x,y) = \underbrace{\sum \sum}_{\text{finite sum}} a_{m,n} e^{i2\pi(mx+ny)},$$

the Parseval relation holds. Since such trigonometric polynomials are dense in $L^2(T^2)$, the Parseval relation holds for every $f \in L^2(T^2)$.

Theorem 9.1 (Minkowski)**.** *Let C be a convex body in R^d of volume V and symmetric about the origin. If $V > 2^d$, then C contains a lattice point other than the origin.*

Proof: We work with $d = 2$. Let C be a convex body in R^2 of volume V and be symmetric about the origin. Assume that C contains no lattice point except the origin. We want to show that the area V of C is ≤ 4.

Let $\phi(x,y)$ be the characteristic function of C. Let

$$f(x,y) = \sum_{m,n} \phi(2(x-m), 2(y-n)).$$

Assume that C is bounded (If C is not bounded, we consider the intersection of C and the circle with center at the origin and radius R. If $V > 4$, then for some large enough R the area of the intersection is > 4. Also note that the intersection is convex, symmetric, and bounded. Thus the Minkowski theorem applies). For each (x,y), this sum has only finitely many nonzero terms.

Let E be the unit square defined as above. The Parseval relation asserts that

$$\sum_{m,n} |a_{m,n}(f)|^2 = \int\int_E |f(x,y)|^2 dxdy.$$

We calculate $a_{m,n}$ as follows:

$$\begin{aligned}
a_{m,n} &= \int\int_E f(x,y) e^{-2\pi i(mx+ny)} dxdy \\
&= \int\int_E \sum_{m,n} \phi(2(x-m), 2(y-n)) e^{-2\pi i(mx+ny)} dxdy \\
&= \int\int_{R^2} \phi(2x, 2y) e^{-2\pi i(mx+ny)} dxdy \\
&= 2^{-d} \int\int_C \phi(x,y) e^{-\pi i(mx+ny)} dxdy.
\end{aligned}$$

The last equality is simply the result of change of variables. For the one above the last equality, we denote by $E_{-m,-n}$ the square with the lower left corner $(-m, -n)$. Then we have:

$$\begin{aligned}
&\int\int_{R^2} \phi(2x, 2y)e^{-2\pi i(mx+ny)}dxdy \\
=& \sum_{m,n}\int\int_{E_{-m,-n}} \phi(2x, 2y)e^{-2\pi i(mx+ny)}dxdy \\
=& \sum_{m,n}\int\int_{E} \phi(2(x-m), 2(y-n))e^{-2\pi i(mx+ny)}dxdy \\
=& \int\int_{E}\sum_{m,n} \phi(2(x-m), 2(y-n))e^{-2\pi i(mx+ny)}dxdy.
\end{aligned}$$

On the other hand, we calculate $\int\int_E |f(x, y)|^2dxdy$.

$$\begin{aligned}
\int\int_E |f(x,y)|^2dxdy &= \int\int_E f(x,y)\sum_{m,n}\phi(2(x-m), 2(y-n))dxdy \\
&= \int\int_{R^2} f(x,y)\phi(2x,2y)dxdy \\
&= \int\int_{R^2}\sum_{m,n}\phi(2(x-m), 2(y-n))\phi(2x,2y)dxdy \\
&= \sum_{m,n}\int\int_{R^2}\phi(2(x-m), 2(y-n))\phi(2x,2y)dxdy \\
&= 2^{-d}\sum_{m,n}\int\int_{R^2}\phi(x-2m, y-2n)\phi(x,y)dxdy \\
&= 2^{-d}\sum_{m,n}\int\int_{C}\phi(x-2m, y-2n)dxdy.
\end{aligned}$$

The Parseval relation gives rise to

$$2^{-2d}|\int\int_C \phi(x,y)e^{-\pi i(mx+ny)}dxdy|^2 == 2^{-d}\sum_{m,n}\int\int_C \phi(x-2m, y-2n)dxdy.$$

If C contains no lattice point except the origin, then one can show that for $(x, y) \in C$ and $(m, n) \neq (0, 0)$, $(x - 2m, y - 2n) \notin C$ and so every term in the sum is zero except the one with $m = n = 0$. The term with $(m, n) = (0, 0)$ equals $2^{-d}V$. Thus we have

$$2^{-d}\sum_{m,n}|\int\int_C \phi(x,y)e^{-\pi i(mx+ny)}dxdy|^2 = V.$$

The term on the left with $(m, n) = (0, 0)$ is $2^{-d}V^2$, and therefore, $2^{-d}V^2 \leq V$, that is, $V \leq 2^d$.

Theorem 9.2. *If C is a convex body in R^d of volume $V = 2^d$ and symmetric about the origin, then there is a lattice point $(m, n) \neq (0, 0)$ in C or on its boundary.*

Proof: Assume that, by a contradiction, $\overline{C}$ contains no lattice point other than the origin. We assume that C is bounded. Thus $\overline{C}$ is compact and there is $\delta > 0$ such that $d(p, C) \geq \delta > 0$ for all lattice points p other than the origin. We may expand $\overline{C}$ slightly to a subset D of R^d so that D is convex and symmetric about origin and yet contains no lattice point other than the origin. Since the volume of D is $> 2^d$, this is in contradiction to Minkowski's theorem.

It remains to show the construction of D. Let

$$D = \{x \in R^d : dist(x, C) \leq \frac{\delta}{2}\}.$$

We claim that if C is (closed) convex, then so is D. Let x and $y \in D$ (Assume that they are not in C. Otherwise, nothing needs to be done.) Let x_0 and $y_0 \in C$ such that $|x - x_0| = dist(x, C)$ and $|y - y_0| = dist(y, C)$. For $0 \leq \lambda \leq 1$, we have $|(\lambda x + (1 - \lambda)y) - (|\lambda x_0 + (1 - \lambda)y_0)| \leq \lambda|x - x_0| + (1 - \lambda)|y - y_0| \leq \frac{\delta}{2}$. Note that C is convex, $\lambda x_0 + (1 - \lambda)y_0 \in C$. Therefore, $dist(\lambda x + (1 - \lambda)y, C) \leq \frac{\delta}{2}$ and so $\lambda x + (1 - \lambda)y \in D$. Clearly, if C is symmetric about the origin then so is D.

Theorem 9.3. *Prove that if $k > 0$ and a, b, c, d are real with $|ad - bc| \leq 1$, then there are integer pairs $(m, n) \neq (0, 0)$ such that*

$$|am + bn| \leq k \qquad |cm + dn| \leq k^{-1}.$$

Deduce that for every real number a, there are infinitely many integer pairs (m, n) such that

$$|a + \frac{n}{m}| \leq \frac{1}{m^2}.$$

Proof: Let

$$C = \{(x, y) : |ax + by| \leq k \text{ and } |cx + dy| \leq \frac{1}{k}\}.$$

Clearly, C is the parallelogram centered at $(0, 0)$ and bounded by $ax+by+k = 0$, $ax+by-k = 0$, $cx+dy+\frac{1}{k} = 0$, and $cx+dy-\frac{1}{k} = 0$. Thus, C is convex and symmetric. Note that C is the set of all points whose distances are $\leq \frac{k}{\sqrt{a^2+b^2}}$ from $ax + by = 0$ and are $\leq \frac{1}{k\sqrt{c^2+d^2}}$ from $cx + dy = 0$. To find the area of C we observe that the mapping $x = du - bv$ and $y = -cu + av$ maps C to the rectangle in the $u - v$ plane bounded by $(ad - bc)u \pm k = 0$ and $(ad - bc)v \pm \frac{1}{k} = 0$. The area of the rectangle is

$$\frac{2y}{|ad - bc|} \cdot \frac{2}{k|ad - bc|}.$$

The Jacobian of the mapping is $|ad - bc|$. Thus

$$V = \frac{2y}{|ad - bc|} \cdot \frac{2}{k|ad - bc|} \cdot |ad - bc| == \frac{4}{|ad - bc|}.$$

If $|ad-bc| \leq 1$ then $V \geq 4$ and so there is a lattice point $(m, n) \neq (0, 0)$ in C or on its boundary. Note that C is closed. The lattice point is in C anyway.

An alternative proof: Let $a \in R$. We want to show there are infinitely many integer pairs m and n such that

$$|a - \frac{n}{m}| \leq \frac{1}{m^2}.$$

Let N be a positive integer. We divide $[0, 1]$ into $N + 1$ parts so that each part has length $\frac{1}{N}$. Consider $\{ja\}$, the fraction part of ja, for $j = 0, 1, \cdots, N$. By pigeonhole principle, there are j and k such that both $\{ja\}$ and $\{ka\}$ lie in some interval of length $\frac{1}{N}$. It follows that there is an integer n so that $|ja - ka - n| \leq \frac{1}{M}$. Let $|j - a| = m$. Then

$$|a - \frac{n}{m}| \leq \frac{1}{mN} \leq \frac{1}{m^2}.$$

10. Measures with bounded powers

Let $M(R)$ be the algebra of complex bounded Borel measures on R. The multiplication of μ and $\nu \in M(R)$ is defined as the convolution $\mu * \nu$, which is the measure in $M(R)$ satisfying

$$\int h(t)d(\mu * \nu)(t) = \int\int h(s + t)d\mu(s)d\nu(t), \qquad \forall h \in C_0(R).$$

The measure $\delta(t)$ with unit mass at 0 is an identity for this algebra. If μ is the measure with mass ϵ, $|\epsilon| = 1$, at x, then ν, defined as the measure with mass $\overline{\epsilon}$ at $-x$, is an inverse of μ. To see this, let $h \in C_0(R)$, we have:

$$\begin{aligned}\int h(t)d(\mu * \nu)(t) &= \int\int h(s + t)d\mu(s)d\nu(t) = \int h(x + t)\epsilon d\nu(t) \\ &= h(0)\epsilon\overline{\epsilon} = h(0) = \int h(t)d\delta(t).\end{aligned}$$

If μ is a point measure with mass ϵ at x then the power μ^{*n} (in the sense of convolution) is the point mass with measure ϵ^n at nx. To see this, let $\nu = \mu^{*(n-1)}$ and $h \in C_0(R)$. We then have

$$\begin{aligned}\int h(t)d(\mu * \nu)(t) &= \int\int h(s + t)d\mu(s)d\nu(t) = \int h(x + t)\epsilon d\nu(t) \\ &= h(x + (n - 1)x)\epsilon\epsilon^{n-1} = h(nx)\epsilon^n = \int h(t)d\mu^{*n}(t).\end{aligned}$$

Also note that if n is negative, then μ^{*n} is defined as

$$\mu^{*n} = (\mu^{-1})^{*|n|}.$$

Of course, in order for this definition to make sense, we must agree that whenever we mention μ^{*n} with negative n, we admit that μ has inverse.

We can identify the set of all bounded point-measures with mass at integers with l^1 by considering such measures as functions on Z. With this correspondence, the identity δ corresponds to $\delta = \{\delta(n)\}_{n=-\infty}^{\infty} \in l^1$ with $\delta(0) = 1$, and $\delta(n) = 0$ for $n \neq 0$, and the convolution of μ and ν is simply the convolution of two elements μ and ν in l^1 defined as

$$(\mu * \nu)(j) = \sum_{k=-\infty}^{\infty} \mu(k)\nu(j-k).$$

When does a given element $\mu \in l^1$ have an inverse? That is, given $\mu \in l^1$, does there exist $\nu \in l^1$ such that $\mu * \nu = \delta$? To answer this question we need Wiener's theorem. By Weiner's theorem we conclude that if $\mu \in l^1$ and $m(x) = \sum \mu(j)e^{ijx}$ is nowhere zero, then there is $\nu \in l^1$ such that $\mu * \nu = \delta$. In fact, the Fourier transform of $\dfrac{1}{m(x)}$ will be the desired ν.

The sequence $a = (a_n) \in l^1$ with $a_0 = 1$ and $a_1 = 1$ has no inverse. In fact, if there were $b \in l^1$ such that $a * b = \delta$, then we would have $\mathcal{F}(a * b) = \mathcal{F}(a) \cdot \mathcal{F}(b) = 1$ with respect to the ordinary multiplication. This is a contradiction, because the two functions are continuous and $\mathcal{F}(a) = 1 + e^{ix}$ equals zero at $x = \pi$, $\mathcal{F}(a) \cdot \mathcal{F}(b) \neq 1$ at $x = \pi$ for any value of $\mathcal{F}(b)$ at $x = \pi$. This shows that if μ has an inverse, then $\mathcal{F}(\mu)$ can be zero nowhere on T.

Moreover, with this correspondence, if μ is the measure with point mass ϵ at p (p is an integer) i.e., $\mu(p) = \epsilon$ and $\mu(k) = 0$ for $k \neq p$, then μ^{*n} is the measure with point mass ϵ^n at np. To see this, let $\nu = \mu^{*(n-1)}$, then

$$\mu^{*n}(k) = \sum_{l=-\infty}^{\infty} \mu(l)\nu(k-l) = \mu(p)\nu(k-p)$$

$= 0$ for all $k \neq np$; ϵ^n for $k = np$.

Let μ be a point measure with mass at integers and let

$$m(x) = \sum \mu(k)e^{ikx}.$$

$m(x)$ is the inverse Fourier transform of $\mu(n)$, and the Fourier coefficient $\int m(x)e^{-ikx}d\sigma(x)$ of $m(x)$ is $\mu(k)$, for $k = 0, \pm 1, \pm 2, \cdots$.

One can show that

$$m^n(x) = \sum \mu^{*n}(k)e^{ikx},$$

where $m^n(x)$ is the nth power of m with respect to ordinary multiplication. We prove this with $n = 2$. Using the Cauchy product of two series, we have

$$\begin{aligned}\int m^2(x)e^{-ikx}d\sigma(x) &= \int(\sum \mu(m)e^{imx})(\sum \mu(n)e^{inx})e^{-ikx}d\sigma(x)\\ &= \int\Big(\sum_n \sum_{l=-\infty}^{\infty} \mu(l)\mu(n-l)e^{inx}\Big)e^{-ikx}d\sigma(x)\\ &= \sum_{l=-\infty}^{\infty} \mu(l)\mu(k-l).\end{aligned}$$

Therefore,

$$m^2(x) = \sum_k\Big(\sum_{l=-\infty}^{\infty} \mu(l)\mu(k-l)\Big)e^{ikx} = \sum_k \mu^{*2}(k)e^{ikx}.$$

Lemma 10.1. *If $f \in l^1$ with $\|f\|_1 \leq K$ and $\|f\|_2 = 1$, then $\|f\|_4 \geq r$, where $r = K^{-1/2} > 0$.*

Proof: For simplicity of notation, we view f as a function defined on R equipped with the measure μ having unit mass at each integer. Let $0 < \theta < 1$ be such that

$$\frac{1}{2} = \frac{1-\theta}{1} + \frac{\theta}{4}.$$

In fact, $\theta = \frac{2}{3}$ will do. Using Hölder's inequality with indices $p = \dfrac{1}{2(1-\theta)}$ and $q = \dfrac{2}{\theta}$, we get

$$(\int_R |f|^2 d\mu)^{1/2} = (\int_R |f|^{2(1-\theta)}|f|^{2\theta}d\mu)^{1/2} \leq (\int_R |f|d\mu)^{1-\theta}(\int |f|^4 d\mu)^{\theta/4}.$$

The desired estimation for $\|f\|_4$ follows.

An alternative proof:

Using Schwarz' inequality twice, we get

$$\begin{aligned}1 &= \|\mu\|_2^2 \leq \||\mu|^{3/2}\|_2\||\mu|^{1/2}\|_2\\ &= \||\mu|^2|\mu|\|_1^{1/2}\|\mu\|_1^{1/2}\\ &\leq (\||\mu|^2\|_2^{1/2}\|\mu\|_2^{1/2})^{1/2}K^{1/2}\\ &= \||\mu|\|_4 K^{1/2}.\end{aligned}$$

Theorem 10.1 (Beurling-Helson)**.** *$\mu \in l^1$ has bounded powers, i.e., $\|\mu^{*n}\|_{l^1} \leq K$ for all $n = 0, \pm 1, \cdots$, if and only if it satisfies $|\mu(p)| = 1$ for some integer p, $\mu(m) = 0$ for all $m \neq p$.*

Proof: (1). Prove that $m(x) = \mathcal{F}^{-1}(\mu)$ can be written as $e^{i\phi(x)}$ for some $\phi(x)$. Reduce the theorem to proving $\phi(x) = px$ for some integer p.

Let $m(x) = \sum \mu(k)e^{ikx}$, $x \in [0, 2\pi)$. Then

$$m^n(x) = \sum \mu^{*n}(k)e^{ikx}.$$

If $\|\mu^{*n}\|_{l^1} \leq K$ for all $n = 0, \pm 1, \cdots$, then $|m^n(x)| \leq K$ for all n and all x. Therefore, $|m(x)| = 1$ for all x. Since $m(x)$ is continuous, we can write

$$m(x) = e^{i\phi(x)},$$

where $\phi(x)$ is continuous and $\phi(x+2\pi) - \phi(x)$ is a (fixed constant) multiple of 2π for all x. To emphasize, we write again that

$$m^n(x) = e^{in\phi(x)} = \sum \mu^{*n}(k)e^{ikx}.$$

The conclusion of the theorem is equivalent to $\phi(x) = px + b$ for some integer p and real number b. Without loss of generality, we assume that $\phi(0) = 0$, and we shall prove $\phi(x) = px$.

(2). Let $\Phi(r, s, t) = \phi(t-r) + \phi(r) - \phi(t-s) - \phi(s)$. Show that $e^{i\Phi(r,s,t)} = w$ for some constant w on a set of positive measure in $[0, 2\pi)^3$. (If $\phi(x) = px$, then $\Phi(r, s, t) = 0$ identically.)

Note that $\sum |\mu^{*n}(k)| \leq K$ (by assumption) and that, by the Parseval relation,

$$\sum_k |\mu^{*n}(k)|^2 = \int |m^n(x)|^2 d\sigma(x) = 1.$$

Therefore, by the lemma, $\sum |\mu^{*n}(k)|^4 \geq r^4 > 0$.

On the other hand, since

$$\mathcal{F}\Big(\int e^{in(\phi(t-s)+\phi(s))}d\sigma(s)\Big) = \left\{(\mu^{*n}(k))^2\right\}_{k\in Z},$$

by the Parseval relation we have

$$\sum |\mu^{*n}(k)|^4 = \int \left| \int e^{in(\phi(t-s)+\phi(s))} d\sigma(s) \right|^2 d\sigma(t).$$

The integral can be written as

$$\begin{aligned}
. \quad & \int \left| \int e^{in(\phi(t-s)+\phi(s))} d\sigma(s) \right|^2 d\sigma(t) \\
= \quad & \int \Big(\int e^{in(\phi(t-s)+\phi(s))} d\sigma(s) \Big) \Big(\int e^{-in(\phi(t-r)+\phi(r))} d\sigma(r) \Big) d\sigma(t) \\
= \quad & \int \int \int e^{in\Phi(r,s,t)} d\sigma(r, s, t),
\end{aligned}$$

where $\Phi(r,s,t) = \phi(t-r) + \phi(r) - \phi(t-s) - \phi(s)$ and $d\sigma(r,s,t) = d\sigma(r)d\sigma(s)d\sigma(t)$. Summarizing, we have

$$\int\int\int e^{in\Phi(r,s,t)} d\sigma(r,s,t) \geq r^4 > 0, \quad \forall n. \qquad (1)$$

Using (1), we proceed to show that $e^{i\Phi(r,s,t)} = w$ for some value w on a set of positive measure. Let $E \subset [0,2\pi)$ and Define

$$\nu(E) = \lambda(\{(r,s,t) \in [0,2\pi)^3 : 0 \leq r,s,t < 2\pi, \Phi(r,s,t) \in E\},$$

where λ is the normalized Lebesgue measure on $[0,2\pi)^3$. Then

$$\int\int\int_{[0,2\pi)^3} e^{in\Phi(r,s,t)} d\sigma(r,s,t) = \int_0^{2\pi} e^{inu} d\nu(u). \qquad (2)$$

Suppose that, by a contradiction, $\Phi(r,s,t)$ assumes each value only on a null set (i.e., there is no subset of $[0,2\pi)^3$ of positive (λ) measure on which Φ takes a constant value). Then, given any point $\theta \in [0,2\pi)$, $\nu(\theta) = 0$ and so ν has no point mass. By Wiener's theorem

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} |\hat{\nu}(k)|^2 = 0.$$

It follows that $\hat{\nu}(k)$ are not bounded away from zero and so that there is a subsequence of $\hat{\nu}(k)$ converging to zero. Since $\hat{\nu}(n)$ is given by the integral on the left hand of (2), by (1) we have $\hat{\nu}(n) \geq r^4$ for all n, and so any subsequence of $\hat{\nu}(n)$ cannot converge to zero. This is a contradiction. Therefore, there is a set of positive measure, denoted by A, such that $e^{i\Phi(r,s,t)} = w$ for some value w on A.

(3). Denote

$$A = \{(r,s,t) \in T^3 : e^{i\Phi(r,s,t)} = w\}.$$

We will show that $A = T^3$.

Let

$$\Psi(r,s,t) = \frac{w^{-1}e^{i\Phi(r,s,t)} + 1}{2}.$$

Then $\Psi = 1$ on A and $|\Psi| < 1$ anywhere else. ($w^{-1}e^{i\Phi}$ is a complex number of modulus 1 and so it is a point on the unit circle and $\dfrac{w^{-1}e^{i\Phi}+1}{2}$ is a point inside the unit disk unless $w^{-1}e^{i\Phi} = 1$). It follows that

$$\Psi^n(r,s,t) \to \chi_A(r,s,t)$$

pointwise as $n \to \infty$.

To prove that $A = T^3$, we show that we are able to extract a subsequence of Ψ^n converging to a continuous function on T^3. Then, since $\Psi^n(r,s,t) \to \chi_A(r,s,t)$, we must have $A = T^3$.

Let $\psi \in l^1(Z^3)$ (sequence depending on three indices) be the Fourier transform of $\Psi(r,s,t)$. Then ψ^{*n} is the Fourier transform of Ψ^n (with respect to ordinary multiplication) and $\|\psi^{*n}\|_{l^1(Z^3)} \le K^4$ for all n.

To see this, we first prove that

$$\|\mathcal{F}(e^{in\Phi(r,s,t)})\|_{l^1(Z^3)} \le K^4, \qquad \forall n.$$

Note that

$$e^{in\Phi(r,s,t)} = e^{in\phi(t-r)}e^{in\phi(r)}e^{-in\phi(t-s)}e^{-in\phi(s)},$$

and that, by (1),

$$\begin{aligned} e^{in\phi(t-r)} &= \sum_j \mu^{*n}(j)e^{-ji(t-r)}, \\ e^{in\phi(r)} &= \sum_j \mu^{*n}(j)e^{-jir}, \\ e^{-in\phi(t-s)} &= \sum_j \overline{\mu^{*n}(j)}e^{-ji(t-s)}, \end{aligned}$$

and

$$e^{-in\phi(s)} = \sum_j \overline{\mu^{*n}(j)}e^{-jis},$$

where each one is viewed as a Fourier series on T^3. Therefore, the Fourier series of $e^{in\Phi(r,s,t)}$ is a (Cauchy) product of all the series above (or the convolution of all Fourier transforms) such that

$$\begin{aligned} \|\mathcal{F}(e^{in\Phi(r,s,t)})\| &= \|\mathcal{F}e^{in\phi(t-r)} * \mathcal{F}e^{in\phi(r)} * \mathcal{F}e^{-in\phi(t-s)} * \mathcal{F}e^{in\phi(t-r)}\| \\ &\le \|\mathcal{F}e^{in\phi(t-r)}\| \cdot \|\mathcal{F}e^{in\phi(r)}\| \cdot \|\mathcal{F}e^{-in\phi(t-s)}\| \cdot \|\mathcal{F}e^{-in\phi(s)}\| \le K^4, \end{aligned}$$

where all the norms are taken in $l^1(Z^3)$.

To see $\|\psi^{*n}\|_{l^1(Z^3)} \le K^4$ for all n, we write

$$\begin{aligned} \Psi^n &= \frac{1}{2^n}[(w^{-1}e^{i\Phi})^n + c(n,1)(w^{-1}e^{i\Phi})^{n-1} + \cdots + 1] \\ &= \frac{1}{2^n}[w^{-n}e^{in\Phi} + c(n,1)w^{-(n-1)}e^{i(n-1)\Phi} + \cdots + 1]. \end{aligned}$$

Therefore,

$$\mathcal{F}(\Psi^n) = \frac{1}{2^n}[w^{-n}\mathcal{F}(e^{in\Phi}) + c(n,1)w^{-(n-1)}\mathcal{F}(e^{i(n-1)\Phi}) + \cdots + 1].$$

It follows, since $|w| = 1$, that

$$\begin{aligned} \|\psi^{*n}\| &= \|\mathcal{F}(\Psi^n)\|_{l^1} \le \frac{1}{2^n}[\|\mathcal{F}(e^{in\Phi})\| + c(n,1)\|\mathcal{F}(e^{i(n-1)\Phi})\| + \cdots + 1 \\ &\le K^4\frac{1}{2^n}[1 + c(n,1) + \cdots + c(n,n-1) + 1] = K^4. \end{aligned}$$

Since $(c_0(Z^3))^* = l^1(Z^3)$, by Alaoglu's theorem, $\{\psi^{*n}\}$ is (sequentially) compact in the weak* topology of $l^1(Z^3)$, that is, there is $\rho \in l^1(Z^3)$ such that for some subsequence ψ^{*n_k}, $\langle \psi^{*n_k}, c\rangle \to \langle \rho, c\rangle$ for any $c \in c_0(Z^3)$ as $n_k \to \infty$.

Note that $|\Psi| \leq 1$ so that $|\Psi^n| \leq 1$ and $\Psi^{n_k} \to \chi_A$ pointwise as $k \to \infty$. By the bounded convergence theorem, we have

$$\begin{aligned} \psi^{*n_k}(m,n,l) &= \mathcal{F}(\Psi^{n_k})(m,n,l) \\ &= \int\int\int_{T^3} \Psi^{n_k}(r,s,t)e^{i(mr+ns+lt)}drdsdt \to \mathcal{F}(\chi_A)(m,n,l), \end{aligned}$$

as $k \to \infty$, that is, $\psi^{*n_k} \to \mathcal{F}(\chi_A)$ componentwise as $k \to \infty$. On the other hand, $\langle \psi^{*n_k}, c\rangle \to \langle \rho, c\rangle$ for any $c \in c_0(Z^3)$ as $k \to \infty$. Taking $c = e^{(m,n,l)}$, we have $\psi^{*n_k}(m,n,l) \to \rho(m,n,l)$, i.e., $\psi^{*n_k} \to \rho$ componentwise. Therefore, $\mathcal{F}(\chi_A) = \rho$ and so

$$\chi_A = \mathcal{F}^{-1}(\rho), \quad a.e.$$

Since the transform $\mathcal{F}^{-1}(\rho)$ of ρ is continuous (because $\rho \in l^1(Z^3)$), $A = T^3$, that is, $e^{i\Phi(r,s,t)}$ assumes value w everywhere on T^3.

(4). Show that $\phi(t) = pt$ for some integer p.

Since $e^{i\Phi(r,s,t)}$ assumes value w everywhere on T^3, $\Phi(r,s,t) = 2k\pi + c$, where $c = arg(w)$ and k is an integer that might depend on r, s, t, apparently. Since Φ is continuous, $\Phi(r,s,t) = 2k\pi + c$ for all r, s, t with a fixed constant k. It follows that $\phi(t-r) + \phi(r)$ is independent of r. Since $\phi(0) = 0$, $\phi(t-r) + \phi(r) = \phi(t)$. In other words,

$$\phi(t+s) = \phi(t) + \phi(s)$$

for all real r, s. Therefore, $m = e^{i\phi}$ is a character of R and, since ϕ is periodic, m is a character of T. Thus, $\phi(t) = pt$ for some integer p.

Homomorphisms of l^1 into l^1

Let h be a homomorphism of l^1 into itself, i.e., h is a linear mapping and $h(\mu * \rho) = h(\mu) * h(\rho)$ for all $\mu, \rho \in l^1$. Note that h is continuous.

Let $e^{(n)}$, $n = 0, \pm 1, \pm 2, \cdots$, be a sequence of elements of l^1 such that $e^{(n)}(n) = 1$ and $e^{(n)}(k) = 0$ for $k \neq n$. Note that $e^{(0)}$ is the identity for the algebra l^1. We have that $h(\mu) = h(e^{(0)}) * h(\mu)$ for all $\mu \in l^1$. Therefore, $h(e^{(0)}) = e^{(0)}$. Let

$$h(e^{(1)}) = \mu \in l^1.$$

Note that $e^{(m)} * e^{(n)} = e^{(m+n)}$. By the multiplicativity of h, $h(e^{(n)}) = \mu^{*n}$, and $\|\mu^{*n}\|_{l^1} \le \|h\|$ (since $\|e^{(n)}\|_{l^1} = 1$) for all n. By Beurling-Helson's Theorem,

$$|\mu(p)| = 1, \text{ for some integer } p, \text{ and} \mu(k) = 0, \text{if } k \neq p.$$

Therefore, we have that

$$\mu = we^{(p)},$$

for some integer p and a complex number w with modulus 1 and that

$$h(e^{(1)}) = we^{(p)}, \quad h(e^{(j)}) = [h(e^{(1)})]^{*j} = \mu^{*j} = w^j e^{(jp)}.$$

Finally, let $\rho \in l^1$. Then, by the linearity and the continuity of ρ,

$$h(\rho) = h(\sum \rho(j)e^{(j)}) = \sum \rho(j)\mu^{*j} = \sum \rho(j)w^j e^{(jp)}.$$

Thus, every homomorphism of l^1 into l^1 has necessarily the above form.

Homomorphisms of $A(T)$ **into** $A(T)$

Let h be a homomorphism of $A(T)$ into itself. Then one can show that h necessarily has a trivial form, i.e., for any $g \in A(T)$,

$$h(g)(e^{it}) = (g \circ m)(e^{it}),$$

where $m(e^{it})$ is the image of e^{it} under h.

In fact,

$$h(g) = \sum \hat{g}(j)h(e^{ijt}) = \sum \hat{g}(j)(m(e^{it}))^j = g(m(e^{it})) \quad (3).$$

That the series can be written as a composition of g and m can be easily seen by replacing e^{it} with $m(e^{it})$ in the Fourier series of g: $g(e^{it}) = \sum \hat{g}(j)e^{ijt} = \sum \hat{g}(j)(e^{it})^j$.

Theorem 10.2. *If $m(e^{it}) \in A(T)$ is any mapping of T into itself such that $g \circ m \in A(T)$ for any $g \in A(T)$, then*

$$m(e^{it}) = we^{ipt}$$

for some integer p and constant w of modulus 1.

Proof: This is a restatement of Beurling- Helson's theorem. If $g \circ m \in A(T)$, then we see from (3) that $|m^j(e^{it})| \le K$ for $j = 0, \pm 1, \cdots$. Let $\mathcal{F}(m) = \mu \in l^1$. Then $\|\mu^{*j}\|_{l^1} \le K$. Using Beurling-Helson's theorem, μ has a special form and so does m.

Let m be any mapping of T into itself. Define the mapping h as

$$h(f) = f \circ m = \sum \hat{f}(j)(m(e^{it}))^j.$$

Obviously, h is a homomorphism of $A(T)$ into $A(T)$. Define a homomorphism of l^1 into l^1, denoted by h', in such a way that

$$h'(\mathcal{F} f) = \mathcal{F}(h(f))$$

for all $f \in A(T)$. This can be written as

$$h'(\rho) = \sum \rho(j)\mu^{*j}$$

for $\rho \in l^1$, where $\mu = \mathcal{F}(m) \in l^1$.

Theorem 10.3. *Assume that m is a mapping of T into T such that $f \circ m \in A(T)$ whenever $f \in A(T)$. Then $h(f) = f \circ m$ is a homomorphism of $A(T)$ into $A(T)$. Define h' as*

$$h'(\mathcal{F} f) = \mathcal{F}(h(f)), \qquad (4)$$

that is, for $\rho \in l^1$,

$$h'(\rho) = \sum_j \rho(j)\mu^{*j},$$

where $\mu = \mathcal{F}(m)$. Then h' is a continuous homomorphism of l^1 into l^1.

Proof: We show that h' is a homomorphism of l^1 into l^1. Let $\rho_k \in l^1$ and let $f_k = \mathcal{F}^{-1}(\rho_k)$, $k = 1, 2$. Then

$$\begin{aligned} h'(\rho_1 * \rho_2) = h'(\mathcal{F} f_1 * \mathcal{F} f_2) &= h'(\mathcal{F}(f_1 \cdot f_2) \\ = \mathcal{F}(h(f_1 \cdot f_2)) &= \mathcal{F}(h(f_1)) * \mathcal{F}(h(f_2)) = h'(\rho_1) * h'(\rho_2). \end{aligned}$$

By the closed graph theorem, it is enough to show that if $\rho^{(n)} \to \rho$ ($l^1(Z)$) and $h'(\rho^{(n)}) \to \rho'$ ($l^1(Z)$), then $\rho' = h'(\rho)$.

$$\begin{aligned} \mathcal{F}^{-1}(\rho') &= \lim_{n\to\infty} \mathcal{F}^{-1}(h'(\rho^{(n)})) \\ &= \lim_{n\to\infty} \mathcal{F}^{-1}(\sum \rho^{(n)}(j)\mu^{*j}) \\ &= \lim_{n\to\infty} (\sum \rho^{(n)}(j) m(e^{it})^j) \\ &= \sum \rho(j) m(e^{it})^j = \sum \rho(j)\mathcal{F}^{-1}(\mu^{*j}) \\ &= \mathcal{F}^{-1}(\sum \rho(j)\mu^{*j}) = \mathcal{F}^{-1}(h'(\rho)). \end{aligned}$$

The first equality holds because $h'(\rho^{(n)}) \to \rho'$ in $l^1(Z)$. The fourth equality holds because $\rho^{(n)} \to \rho$ ($l^1(Z)$) and $|m(e^{it})| \leq 1$.

Theorem 10.4. *Let h' be any homomorphism of l^1 into l^1. Then there exists $m : T \to T$ so that for any $\rho \in l^1$,*

$$\mathcal{F}^{-1}[h'(\rho)] = \mathcal{F}^{-1}(\rho) \circ m.$$

It follows that any homomorphism of l^1 into l^1 is of the form (4), where $h(f) = f \circ m$.

Proof: For each $q \in T$, define

$$H_q(f) = [\mathcal{F}^{-1}(h'(\rho))](q),$$

where $\rho = \mathcal{F}(f)$. Then H_q is a homomorphism of $A(T)$ into C. In fact, if we denote $\rho = \mathcal{F} f$ and $\eta = \mathcal{F} g$, then

$$\begin{aligned} H_q(f \cdot g) &= [\mathcal{F}^{-1}(h'(\rho * \eta))](q) \\ &= [\mathcal{F}^{-1}(h'(\rho) * h'(\eta))](q) \\ &= [\mathcal{F}^{-1}(h'(\rho)) \cdot \mathcal{F}^{-1}(h'(\eta))](q) \\ &= H_q(f) \cdot H_q(g). \end{aligned}$$

Moreover, by the Gelfand theorem, H_q must be continuous. Therefore, H_q is an evaluation of the form $H_q(f) = f(q')$ for some $q' \in T$. Define $m : T \to T$ such that $q' = m(q)$. Then, for any $\rho \in l^1$, let $f = \mathcal{F}^{-1}(\rho)$, we have

$$[\mathcal{F}^{-1}(h'(\rho))](q) = H_q(f) = f(q') = f \circ m(q)$$

for all $q \in T$.

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